

Orthogonal basis for the Shapovalov form on $U_q(\mathfrak{sl}(n+1))$

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Abstract

Let U be either classical or quantized universal enveloping algebra of the Lie algebra $\mathfrak{sl}(n+1)$ extended over the field of fractions of the Cartan subalgebra. We suggest a PBW basis in U over the extended Cartan subalgebra diagonalizing the contravariant Shapovalov form on generic Verma module. The matrix coefficients of the form are calculated and the inverse of the form is explicitly constructed.

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1 Introduction

The contravariant bilinear form on Verma modules is a fundamental object in the representation theory of simple complex Lie algebras and quantum groups, which is responsible for many important properties including irreducibility, [1]. Its inverse is closely related with intertwining operators [2], the dynamical Yang-Baxter equation [3], and invariant star product on homogeneous spaces, [4, 5].

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Contravariant forms on highest weight modules descend from a bilinear form on the universal enveloping algebra with values in the Cartan subalgebra. It was introduced and studied by Shapovalov [6], who computed the determinant for its restriction to every weight subspace. It was extended to quantum groups in [7]. The determinant formula was further generalized for parabolic Verma modules over the classical universal enveloping algebras in [8]. These results provided a criterion for the corresponding modules to be irreducible, since the kernel of a contravariant form is invariant.

Applications to mathematical physics require the knowledge of the inverse Shapovalov form, which explicit expression is an open problem for general simple Lie algebras. The most important advance in this direction was made in [9], where matrix coefficients of the pairing on Mickelsson algebras were calculated. However, [9] does not address the Verma modules focusing on different problem. Although the inverse Shapovalov form for the A_n series can be derived from [9], a self-contained presentation is still missing in the literature. In the present paper we give an independent elementary derivation based on the definition of the quantum group. We construct the orthogonal basis of the Shapovalov form on $U_q(\mathfrak{gl}(n+1))$ and obtain a similar result for $U(\mathfrak{gl}(n+1))$ via the classical limit. Of course, the classical case can be done directly, in an even simpler way. The ground field is fixed to \mathbb{C} but can be changed to an arbitrary field of zero characteristic.

We consider a system of "dynamical root vectors" $\hat{e}_{\pm\mu}$ in the Borel subalgebras. Upon appropriate ordering, it gives rise to a Poincaré-Birkhoff-Witt (PBW) basis over the (extended) Cartan subalgebra. The vectors $\hat{e}_{\pm\mu}$ are constructed from the Chevalley generators through generalized commutators with coefficients in the Cartan subalgebra. The positive and negative dynamical root vectors are related via $\omega(\hat{e}_{\pm\mu}) = \hat{e}_{\mp\mu}$, where ω is the anti-algebra Chevalley involution. This PBW system diagonalizes the Shapovalov form on every Verma module M_λ and is complete if the highest weight λ is away from a family of hyperplanes. This family is wider than the zero set of the Shapovalov determinant, which is known to be $\cup_{\alpha \in R^+} \{\lambda | (\lambda + \rho, \alpha) \in \mathbb{N}\}$ for $U(\mathfrak{g})$ and $\cup_{\alpha \in R^+} \{\lambda | q^{2(\lambda + \rho, \alpha)} \in q^{2\mathbb{N}}\}$ for $U_q(\mathfrak{g})$. Our set of singular points is still contained in $\cup_{\alpha \in R^+} \{\lambda | (\lambda, \alpha) \in \mathbb{Z}\}$ for $U(\mathfrak{g})$ and in $\cup_{\alpha \in R^+} \{\lambda | q^{2(\lambda, \alpha)} \in q^{2\mathbb{Z}}\}$ for $U_q(\mathfrak{g})$. Away from this set, the dynamical PBW system is a basis. We compute the matrix coefficients and construct the inverse form for generic weight, off the union of hyperplanes where some of the matrix coefficients vanish.

The dynamical root vectors project to generators of the Mickelsson algebras associated with a chain of subalgebras $\mathfrak{sl}(i) \subset \mathfrak{sl}(i+1)$, $i = 2, \dots, n$. Essentially they are raising and lowering operators participating in construction of the Gelfand-Zetlin basis in finite

dimensional $U_q(\mathfrak{g})$ -modules, [10]. Elements of the Gelfand-Zetlin basis are formed by common eigenvectors of the commutative subalgebra generated by $U_q(\mathfrak{h})$ and the center of $U_q(\mathfrak{sl}(i))$, $i = 2, \dots, n+1$. The dynamical PBW monomials feature the same property and form the Gelfand-Zetlin basis in Verma modules.

The paper is organized as follows. After the preliminary section containing the basics on the quantum group $U_q(\mathfrak{sl}(n+1))$, we introduce the dynamical root vectors and study their key properties. Then we show that, upon an appropriate ordering, the systems of positive and negative dynamical PBW monomials give rise to dual bases in right lower and left upper Verma modules with respect to the cyclic Shapovalov pairing. We compute the matrix coefficients and construct the inverse of the cyclic form. Further we pass from the cyclic form to contravariant and prove that the PBW system of negative dynamical root vectors yields an orthogonal basis. This should be regarded as a refinement of the cyclic result and it is based on a "row-wise commutativity" of dynamical root vectors proved therein. Further we illustrate the key steps on the example of A_2 . In the last section, we apply the dynamical root vectors to construction of singular vectors in the Verma modules.

2 Preliminaries: the quantum group $U_q(\mathfrak{sl}(n+1))$

For a guide in quantum groups, the reader is referred to [1] or [11], or to the original paper [12]. In this section we collect the facts about quantum $\mathfrak{sl}(n+1)$ that are relevant to this exposition.

Let us fix some general notation. We work over the ground field \mathbb{C} of complex numbers. By \mathbb{Z} we denote the set of all integers, by \mathbb{Z}_+ the subset of non-negative and by \mathbb{N} the subset of strictly positive integers. Given $a, b \in \mathbb{Z}$ we understand by $[a, b] \subset \mathbb{Z}$ the interval of all integers from a to b inclusive. We also use the notation $(a, b]$, $[a, b)$, and (a, b) for intervals without one or two boundaries.

Throughout the paper, \mathfrak{g} stands for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n+1)$, $n \geq 1$. The case $n = 1$ is trivial, and we are mostly interested in $n \geq 2$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let $R \subset \mathfrak{h}^*$ denote the root system of \mathfrak{g} with a subsystem R^+ of positive roots, relative to \mathfrak{h} . The choice of R^+ facilitates a triangular decomposition, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where \mathfrak{n}^\pm are nilpotent Lie subalgebras corresponding to the positive and negative roots. Let (\cdot, \cdot) designate the canonical inner product on \mathfrak{h}^* .

Denote by $\Pi^+ \subset R^+$ the basis of simple positive roots $\{\alpha_1, \dots, \alpha_n\}$, with the standard ordering determined up to the inversion by the condition $(\alpha_i, \alpha_j) = 0$ for $|i - j| > 1$.

For any pair of integers $i, j \in [1, n]$ such that $i \leq j$ let $\mathfrak{g}_{ij} \subset \mathfrak{g}$ be the Lie subalgebra $\mathfrak{sl}(j - i + 2)$ corresponding to the roots $\alpha_i, \dots, \alpha_j \in \Pi^+$. We also consider the Cartan subalgebra $\mathfrak{h}_{ij} = \mathfrak{g}_{ij} \cap \mathfrak{h}$ and nilpotent subalgebras $\mathfrak{n}_{ij}^\pm = \mathfrak{g}_{ij} \cap \mathfrak{n}^\pm$, so that $\mathfrak{g}_{ij} = \mathfrak{n}_{ij}^- \oplus \mathfrak{h}_{ij} \oplus \mathfrak{n}_{ij}^+$ is a triangular decomposition compatible with the decomposition of \mathfrak{g} .

We assume that $q \in \mathbb{C}$ is not a root of unity and define $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$ for an indeterminate x . The quantum group $U_q(\mathfrak{g})$ is a \mathbb{C} -algebra generated by $e_i, f_i, t_i^{\pm 1}$, $i \in [1, n]$, subject to the Chevalley relations

$$t_i e_j = q^{(\alpha_i, \alpha_j)} e_j t_i, \quad t_i f_j = q^{-(\alpha_i, \alpha_j)} f_j t_i, \quad [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}},$$

and the Serre relations

$$e_i^2 e_j - [2]_q e_i e_j e_i + e_j e_i^2 = 0, \quad f_i^2 f_j - [2]_q f_i f_j f_i + f_j f_i^2 = 0, \quad |i - j| = 1,$$

$$[e_i, e_j] = 0, \quad [f_i, f_j] = 0, \quad |i - j| > 1.$$

The elements e_i and f_i are called, respectively, the positive and negative Chevalley generators. The assignment $\omega: t_i \mapsto t_i$, $\omega: e_i \mapsto f_i$, $\omega: f_i \mapsto e_i$ extends to an anti-algebra automorphism of $U_q(\mathfrak{g})$ called Chevalley involution.

The quantum group can be also defined as an algebra over the ring of fractions of $\mathbb{C}[q, q^{-1}]$ over the multiplicative system generated by $q^m - 1$, $m \in \mathbb{N}$. Its $\mathbb{C}[[\hbar]]$ -version is a $\mathbb{C}[[\hbar]]$ -extension of $U_q(\mathfrak{g})$ completed in the \hbar -adic topology. The extension goes through the embedding $\mathbb{C}[q, q^{-1}] \rightarrow \mathbb{C}[[\hbar]]$, $q \mapsto e^\hbar$. The corresponding relations translate to

$$[h_i, e_j] = (\alpha_i, \alpha_j) e_j, \quad [h_i, f_j] = -(\alpha_i, \alpha_j) f_j, \quad [e_i, f_j] = \delta_{ij} [h_i]_q.$$

upon the substitution $t_i^{\pm 1} = q^{\pm h_i}$. This algebra, denoted by $U_\hbar(\mathfrak{g})$, is a deformation of the classical universal enveloping algebra $U(\mathfrak{g})$. It is still convenient to use the notation $[h_i]_q = \frac{t_i - t_i^{-1}}{q - q^{-1}}$ and $q^{h_i} = t_i$ when working with $U_q(\mathfrak{g})$. This makes sense of $[h]_q \in U_q(\mathfrak{g})$ for any linear combination $h = c_0 + \sum_{i=1}^n c_i h_i$ with integer c_i , $i > 0$, and arbitrary complex c_0 . We denote by $U_q(\mathfrak{h})$ the subalgebra in $U_q(\mathfrak{g})$ generated by $\{t_i^{\pm 1}\}_{i=1}^n$. This q -version of the Cartan subalgebra is the polynomial ring on a torus, while $U(\mathfrak{h})$ is a polynomial ring on a vector space. Note that $\mathfrak{h} \not\subset U_q(\mathfrak{h})$ contrary to $U_\hbar(\mathfrak{h})$, which stands for the subalgebra in $U_\hbar(\mathfrak{g})$ generated by $\{h_i\}_{i=1}^n$.

Observe that $U_q(\mathfrak{g}_{ij})$ is a natural subalgebra in $U_q(\mathfrak{g})$ for any pair $i, j \in [1, n]$ such that $i \leq j$. Here are other subalgebras of importance in $U_\hbar(\mathfrak{g})$. The elements e_i and f_i generate, respectively, the subalgebras $U_q(\mathfrak{n}^+)$ and $U_q(\mathfrak{n}^-)$. Their $\mathbb{C}[[\hbar]]$ -extensions $U_\hbar(\mathfrak{n}^\pm)$

are deformations of the classical universal enveloping algebras $U(\mathfrak{n}^\pm)$. The quantum Borel subalgebras $U_q(\mathfrak{b}^\pm)$ are generated by $U_q(\mathfrak{n}^\pm)$ over $U_q(\mathfrak{h})$.

All positive roots in R^+ are sums $\alpha_i + \dots + \alpha_j$, where $i \leq j$. Put $e_{ii} = e_i$, $f_{ii} = f_i$ and extend this definition inductively by

$$e_{ij} := [e_{i+1j}, e_i]_q, \quad f_{ij} := [f_i, f_{i+1j}]_q$$

for $i < j$. Here $[x, y]_q$ is the generalized commutator $xy - qyx$. Along with e_{ii} , f_{ii} we will also use the usual notation e_i , f_i . Note that the positive and negative root vectors are related via the Chevalley involution, $\omega(f_{ij}) = e_{ij}$.

We define \mathfrak{n}^\pm in the q -case as the linear spans $\mathfrak{n}^+ = \{e_{km}\}_{k \leq m} \subset U_q(\mathfrak{g})$ and $\mathfrak{n}^- = \{f_{km}\}_{k \leq m} \subset U_q(\mathfrak{g})$. These are $U_h(\mathfrak{h})$ -submodules, which are trivial deformations of the classical $U(\mathfrak{h})$ -modules $\mathfrak{n}^\pm \subset \mathfrak{g}$. Similarly, we put $\mathfrak{n}_{km}^\pm = \mathfrak{n}^\pm \cap U_q(\mathfrak{g}_{km})$, so that $\mathfrak{n}_{km}^+ = \text{Span}\{e_{ij}\}_{k \leq i \leq j \leq m}$ and $\mathfrak{n}_{km}^- = \text{Span}\{f_{ij}\}_{k \leq i \leq j \leq m}$.

Lemma 2.1. *Suppose that $k \in (i, j) \subset [1, n]$. Then $[f_k, f_{ij}] = 0 = [e_k, f_{ij}]$. Further, $[f_{ij}, f_i]_q = 0$, $[f_j, f_{ij}]_q = 0$, $[e_i, f_{ij}] = f_{i+1j}q^{-h_i}$, $[e_j, f_{ij}] = -qf_{ij-1}q^{h_i}$.*

Proof. Direct calculation. □

In what follows, we deal with a general algebraic concept, which we recall here. Consider a unital associative algebra A and a non-empty subset $I \subset A$. Let AI denote the left ideal generated by I . We denote by A^I the subset of elements $x \in A$ such that $Ix \subset AI$. We write simply A^a when $I = \{a\}$ consists of one element a . Obviously A^I is not empty, $A^I \supset AI$, and is a subalgebra in A . It is the normalizer of AI , i.e. the maximal subalgebra in A where AI is a two-sided ideal. For every $x \in A^I$ the map $x: AI \rightarrow AIx \subset AI$ amounts to an anti-homomorphism $A^I \mapsto \text{End}_A(AI)$, where the ideal AI is regarded as a natural submodule of the regular left A -module. Obviously $x \in A^I$ if and only if $[x, I] \subset AI$.

Similarly one defines the normalizer IA of the right ideal IA generated by I . As in the left case, $x \in {}^IA$ if and only if $[x, I] \subset IA$.

In our setting, A will be $U := U_q(\mathfrak{g})$. If I a subset of simple positive root vectors and \mathfrak{g}' is the corresponding reductive subalgebra in \mathfrak{g} , the quotient A^I/AI is the Mickelsson algebra $S(\mathfrak{g}, \mathfrak{g}')$, [13].

We finish our introduction to the quantum special linear group with two lemmas that will be used in what follows. Let S_m denote the symmetric group of permutations of m symbols.

Lemma 2.2. *Suppose that $m \in [2, n]$. For any $\sigma \in S_m$, the Chevalley monomial $f_{\sigma(1)} \dots f_{\sigma(m)}$ belongs to ${}^{\mathfrak{n}_{2m}}U$. Moreover, $f_{\sigma(1)} \dots f_{\sigma(m)} \in \mathfrak{n}_{2m}^-U$, provided $\sigma \neq \text{id}$.*

Proof. Consider the case $\sigma = \text{id}$ first, using induction on m . For $m = 2$ the statement immediately follows from the Serre relations: $(f_1 f_2) f_2 = f_2 ([2]_q f_1 f_2 - f_2 f_1) \in \mathfrak{n}_{2m}^- U$, while the second statement is obvious. Suppose that $m > 2$ and the lemma has been proved for all i from the interval $[2, m)$. Then, for such i , the Serre relations give

$$f_1 \dots f_m f_i = f_1 \dots f_i f_{i+1} f_i \psi = \frac{1}{[2]_q} f_1 \dots f_{i-1} f_i^2 f_{i+1} \psi + \frac{1}{[2]_q} f_1 \dots f_{i-1} f_{i+1} f_i^2 \psi,$$

where $\psi = f_{i+2} \dots f_m$ and $\psi = 1$ if $i = m - 1$. By the induction assumption, the first term belongs to $\mathfrak{n}_{2i}^- U$. In the second term, f_{i+1} commutes with $f_1 \dots f_{i-1}$. Therefore, the second term belongs to $f_{i+1} U \subset \mathfrak{n}_{2m}^- U$, and the sum lies in $\mathfrak{n}_{2m}^- U$. For $i = m$, we have

$$f_1 \dots f_m f_m = [2]_q f_1 \dots f_{m-2} f_m f_{m-1} f_m - f_1 \dots f_{m-2} f_m^2 f_{m-1} \in f_m U \subset \mathfrak{n}_{2m}^- U.$$

This proves the statement for all m and $\sigma = \text{id}$.

Suppose that $\sigma \neq \text{id}$. The statement is obvious if $\sigma(1) \neq 1$. Otherwise let $i \in [2, m)$ be the least integer such that $\sigma(i) \neq i$. Put $\psi = f_{\sigma(i+1)} \dots f_{\sigma(m)}$. Then

$$f_{\sigma(1)} \dots f_{\sigma(m)} = f_1 \dots f_{i-1} f_{\sigma(i)} \psi = f_{\sigma(i)} f_1 \dots f_{i-1} \psi \in \mathfrak{n}_{2m}^- U,$$

as $\sigma(i) > i \geq 2$. This proves the statement for $\sigma \neq \text{id}$. \square

Lemma 2.3. *Suppose that integers $i, j, k, m \in [1, n]$ satisfy the inequalities $i \leq j \leq k < m$. Then for all $u \in U_q(\mathfrak{n}_{ik}^+)$, $[u, f_{jm}] \in \mathfrak{n}_{j+1m}^- U$.*

Proof. Introduce a grading in $U_q(\mathfrak{n}^+)$ setting $\deg e_j = 1$ for all $j \in [1, n]$. Let $u \in U_q(\mathfrak{n}_{ik}^+)$ be a Chevalley monomial. The statement is trivial for zero degree u , so we assume $\deg u > 0$. Present u as a product $u = u' e_l$ for some $e_l, u' \in U_q(\mathfrak{n}_{ik}^+)$. If $\deg u' = 0$ and $u = e_l$, then the statement follows from the formula $[e_l, f_{jm}] = \delta_{jl} f_{j+1m} q^{-h_j}$, cf. Lemma 2.1. For $\deg u \geq 1$, induction on $\deg u$ gives

$$u f_{jm} = u' f_{jm} e_l + \delta_{jl} u' f_{j+1m} q^{-h_j} \in f_{jm} u' e_l + \mathfrak{n}_{j+1m}^- U + \delta_{jl} f_{j+1m} u' q^{-h_j} + \mathfrak{n}_{j+2m}^- U,$$

where the last summand is present only if $j + 2 \leq m$. The right-hand side is contained in $f_{jm} u + \mathfrak{n}_{j+1m}^- U$, as required. \square

3 Dynamical root vectors

We set up an ordering on positive root vectors e_{ij} induced by the lexicographic ordering on pairs (i, j) , $i \leq j$. The negative root vectors f_{ij} are ordered in the opposite way. These

orderings are normal and compatible with a reduced decomposition of the maximal element in the Weyl group of \mathfrak{g} . The ordered systems of root vectors generate a PBW basis in the algebras $U_q(\mathfrak{n}^\pm)$, [11]. The Shapovalov form, which is the subject of our interest, is very complicated in this basis. We need a new basis suitable for our study, possibly on the extension of $U_q(\mathfrak{g})$ over the ring of fractions of $U_q(\mathfrak{h})$ over some multiplicative system. This basis is introduced in this section.

Put $h_{ik} := h_i + \dots + h_k + k - i$ for all positive integer i, k such that $i \leq k$. The difference $k - i$ is equal to $(\rho, \alpha_{ik}) - 1$, where $\alpha_{ik} = \alpha_i + \dots + \alpha_k \in R^+$. We define dynamical root vectors $\hat{f}_{ik} \in U_q(\mathfrak{b}^-)$ and $\hat{e}_{ik} \in U_q(\mathfrak{b}^+)$ for all pairs $i, k \in [1, n]$ of integers such that $i \leq k$. For $k = i \in [1, n]$ we put $\hat{e}_{ii} = e_i$ and $\hat{f}_{ii} = f_i$. For $i < k$ we proceed recursively by

$$\hat{e}_{ik} = q^{-1}[h_{i+1k}]_q[\hat{e}_{i+1k}, e_i]_q + q^{h_{i+1k}}\hat{e}_{i+1k}e_i, \quad \hat{f}_{ik} = q^{-1}[f_i, \hat{f}_{i+1k}]_q[h_{i+1k}]_q + f_i\hat{f}_{i+1k}q^{h_{i+1k}},$$

The right-hand side can be expressed through "generalized commutators" with coefficients from the Cartan subalgebra. For instance,

$$\hat{e}_{ik} = [h_{i+1k} + 1]_q\hat{e}_{i+1k}e_i - [h_{i+1k}]_qe_i\hat{e}_{i+1k}, \quad \hat{f}_{ik} = f_i\hat{f}_{i+1k}[h_{i+1k} + 1]_q - \hat{f}_{i+1k}f_i[h_{i+1k}]_q.$$

Note that the Cartan coefficients in \hat{e}_{i+1k} commute with e_i and can be gathered on the left. Similarly to the standard root vectors, $\omega(\hat{e}_{ij}) = \hat{f}_{ij}$.

The name dynamical follows the analogy with the dynamical Yang-Baxter equation from the mathematical physics literature, [3]. In a representation, the Cartan coefficients are specialized at the weight of a particular vector the elements \hat{e}_{ij} and \hat{f}_{ij} act upon. This dependence on the weight is "dynamical" rather than "statical" since the Cartan coefficients are not central in $U_q(\mathfrak{g})$.

The key properties of dynamical root vectors are described by the following proposition.

Proposition 3.1. *For all integer $i, j, k \in [1, n]$ such that $i < j, k$,*

$$\hat{f}_{ij} \in U^{e_k}, \quad [e_i, \hat{f}_{ij}] = \hat{f}_{i+1j}[h_{ij}]_q \mod Ue_i,$$

$$\hat{e}_{ij} \in {}^{f_k}U, \quad [\hat{e}_{ij}, f_i] = [h_{ij}]_q\hat{e}_{i+1j} \mod f_iU.$$

Proof. We will check only the first line. The second line is obtained from it via the Chevalley involution.

It is obvious that $\hat{f}_{ij} \in U^{e_k}$ for $k > j$, so we assume $i < k \leq j$. For $j = i + 1$ we have $[e_j, \hat{f}_{ij}] = [f_i, [h_j]_q]q^{-1}[h_j]_q + f_i[h_j]_qq^{h_j}$ modulo Ue_j . The retained terms give

$$f_i((h_j]_q - q[h_j + 1]_q)q^{-1}[h_j]_q + q^{h_j}[h_j]_q = 0,$$

hence $[e_j, \hat{f}_{ij}] \in Ue_j$, as required. For the right equality in the first line, we have

$$[e_i, \hat{f}_{ij}] = [[h_i]_q, f_j]_q q^{-1} [h_j]_q + [h_i]_q f_j q^{h_j} + \dots = f_j q^{-h_i} q^{-1} [h_j]_q + f_j [h_i + 1]_q q^{h_j} \dots$$

where we have omitted the terms from Ue_i . Modulo those terms, the last expression is equal to $f_j [h_i + h_j + 1]_q = f_j [h_{ij}]_q$ for $j = i + 1$. This proves the proposition for $j = k = i + 1$.

Further we do induction on $j - i$. The case $j - i = 1$ is already done. Suppose that the proposition is proved for $j - i$ up to $l - 1 > 0$. Then $[e_k, \hat{f}_{i+1j}] \in Ue_k$ for $k \in [i + 2, j]$. This immediately implies the inclusion $[e_k, \hat{f}_{ij}] \in Ue_k$ for such k , thanks to the recursive presentation of \hat{f}_{ij} through \hat{f}_{i+1j} . For $k = i + 1$ we have

$$[e_k, \hat{f}_{ij}] = [f_i, [e_k, \hat{f}_{kj}]]_q q^{-1} [h_{kj}]_q + f_i [e_k, \hat{f}_{kj}] q^{h_{kj}} + \dots,$$

where the omitted terms lie in Ue_k . By the induction assumption, the remaining terms give

$$[f_i, \hat{f}_{k+1j} [h_{kj}]_q]_q q^{-1} [h_{kj}]_q + f_i \hat{f}_{k+1j} q^{h_{kj}} [h_{kj}]_q,$$

up to the terms from Ue_{i+1} . This is equal to the product of $f_i \hat{f}_{k+1j}$ (observe that f_i commutes with $\hat{f}_{k+1j} = \hat{f}_{i+2j}$) and the Cartan factor

$$[h_{kj}]_q (([h_{kj}]_q - q[h_{kj} + 1]_q) q^{-1} + q^{h_{kj}}) = 0.$$

Therefore, $[e_{i+1}, \hat{f}_{ij}] \in Ue_{i+1}$, as required.

To complete the induction, we need to check the rightmost equality:

$$\begin{aligned} [e_i, \hat{f}_{ij}] &= [[h_i]_q, \hat{f}_{i+1j}]_q q^{-1} [h_{i+1j}]_q + [h_i]_q \hat{f}_{i+1j} q^{h_{i+1j}} + \dots \\ &\quad \hat{f}_{i+1j} \left(([h_i + 1]_q - q[h_i]_q) q^{-1} [h_{i+1j}]_q + [h_i + 1]_q q^{h_{i+1j}} \right) + \dots \end{aligned} \quad (3.1)$$

where we have dropped the terms from Ue_i . The Cartan factor in the brackets is

$$q^{-h_i} q^{-1} [h_{i+1j}]_q + [h_i + 1]_q q^{h_{i+1j}} = [h_{ij}]_q.$$

This completes the induction on $l = j - i$ and the proof of the proposition. \square

Let $h_\alpha \in \mathfrak{h}$ denote the element determined by $\alpha(h_\alpha) = (\lambda, \alpha)$ for all $\lambda \in \mathfrak{h}^*$. Consider the multiplicative system in $U_q(\mathfrak{h})$ generated by $[h_\alpha + m]_q$, $\alpha \in R^+$, $m \in \mathbb{Z}$, and denote by $\hat{U}_q(\mathfrak{h})$ the ring of fractions of $U_q(\mathfrak{h})$ over this system. One can check that there is a natural extension, $\hat{U}_q(\mathfrak{g})$, of $U_q(\mathfrak{g})$ over $\hat{U}_q(\mathfrak{h})$. The algebra $\hat{U}_q(\mathfrak{g}_{ij})$ contains an idempotent p_{ij} of zero weight such that $p_{ij} \hat{U}_q(\mathfrak{g}) = \{x \in \hat{U}_q(\mathfrak{g}) : \mathfrak{n}_{ij}^+ x = 0\}$, $\hat{U}_q(\mathfrak{g}) p_{ij} = \{x \in \hat{U}_q(\mathfrak{g}) : x \mathfrak{n}_{ij}^- = 0\}$, [14, 15]. It is called extremal projector of the subalgebra $\hat{U}_q(\mathfrak{g}_{ij})$.

Proposition 3.2. *The vector $\hat{f}_{ij}p$ is equal to $pf_{ij} \prod_{l=i+1}^j [h_{lj} + 1]_q$, where $p = p_{i+1j}$.*

Proof. By construction, $\hat{f}_{ij}p$ belongs to $pU_q(\mathfrak{b}^-)$ and hence to $pU_q(\mathfrak{b}^-)p$. By Lemma 2.2, pf_{ij} belongs to $U_q(\mathfrak{b}^-)p$ and hence to $pU_q(\mathfrak{b}^-)p$. On the other hand, $pf_{ij}p = pf_{\alpha_i} \dots f_{\alpha_j}p$ is a unique, up to a scalar factor, vector of weight α_{ij} in $pU_q(\mathfrak{g}_-)p$. Now observe that $f_{\alpha_i} \dots f_{\alpha_j}$ enters \hat{f}_{ij} with the Cartan coefficient $\prod_{l=i+1}^j [h_{lj} + 1]_q$. \square

It follows that \hat{e}_{ij} and \hat{f}_{ij} generate a PBW basis in $\hat{U}_q(\mathfrak{g})$ over $\hat{U}_q(\mathfrak{h})$.

4 Verma modules

Thanks to a PBW basis, the algebra $U_q(\mathfrak{g})$ is a free $U_q(\mathfrak{n}^-) - U_q(\mathfrak{n}^+)$ -bimodule generated by $U_q(\mathfrak{h})$. The triangular factorization $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-)U_q(\mathfrak{h})U_q(\mathfrak{n}^+)$ gives rise to the direct sum decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{h}) \oplus [\mathfrak{n}^-U_q(\mathfrak{g}) + U_q(\mathfrak{g})\mathfrak{n}^+]$, which facilitates a projection $\pi: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$. The Shapovalov form is a linear mapping $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$, defined as the composition

$$U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \xrightarrow{\omega \otimes \text{id}} U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \xrightarrow{\pi} U_q(\mathfrak{h}),$$

where the middle arrow is the multiplication. The form is ω -contravariant, i.e. the conjugation operation factors through ω . The left ideal $U_q(\mathfrak{g})\mathfrak{n}^+$ lies in the kernel of the form, which therefore restricts to the quotient $U_q(\mathfrak{g})/U_q(\mathfrak{g})\mathfrak{n}^+$.

It is convenient to drop the extra structure of Chevalley involution and consider pairings between left and right modules, with cyclicity in place of contravariance. Recall that a pairing $\langle \cdot, \cdot \rangle: V \otimes W$ between a right module V and left module W is called cyclic if $\langle xu, y \rangle = \langle x, uy \rangle$ for all $x \in V$, $u \in W$, and $u \in U_q(\mathfrak{g})$. Specifically the cyclic Shapovalov form is defined similarly to contravariant but without the first arrow. It induces a cyclic pairing between the right and left quotient modules $\mathfrak{n}^-U_q(\mathfrak{g}) \backslash U_q(\mathfrak{g})$ and $U_q(\mathfrak{g})/U_q(\mathfrak{g})\mathfrak{n}^+$.

The Shapovalov form on $U_q(\mathfrak{g})$ is equivalent to a family of forms on Verma modules parameterized by the highest weight $\lambda \in \mathfrak{h}^*$. Consider a one dimensional representation of the Cartan subalgebra $U_q(\mathfrak{h})$ determined by the assignment $t_i \mapsto q^{\lambda_i} \in \mathbb{C}$, where $\lambda_i = (\lambda, \alpha_i)$. It extends to a representation of $U_q(\mathfrak{b}^\pm)$ by letting $\lambda(\mathfrak{n}^\pm) = 0$. We regard \mathbb{C} as a left $U_q(\mathfrak{b}^+)$ -module and right $U_q(\mathfrak{b}^-)$ -module with respect to these extensions and denote it by \mathbb{C}_λ . Define the right and left Verma $U_q(\mathfrak{g})$ -modules M_λ^* and M_λ to be the induced modules

$$M_\lambda^* = \mathbb{C}_\lambda \otimes_{U_q(\mathfrak{b}^-)} U_q(\mathfrak{g}), \quad M_\lambda = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}^+)} \mathbb{C}_\lambda,$$

When restricted to the Cartan subalgebra, M_λ^\star is isomorphic to $\mathbb{C}_\lambda \otimes U_q(\mathfrak{n}^+)$, while M_λ is isomorphic to $U_q(\mathfrak{n}^+) \otimes \mathbb{C}_\lambda$. Denote by $v_\lambda^\star \in M_\lambda^\star$ and $v_\lambda \in M_\lambda$ their canonical generators. They carry the highest weights.

The cyclic Shapovalov pairing $M_\lambda^\star \otimes M_\lambda \rightarrow \mathbb{C}$ is defined by

$$\langle v_\lambda^\star x, y v_\lambda \rangle = \lambda(\pi(xy)), \quad x, y \in U.$$

By construction, it is normalized to $\langle v_\lambda^\star, v_\lambda \rangle = 1$ and it is a unique cyclic pairing between M_λ^\star and M_λ that satisfies this condition. In order to simplify formulas, we suppress the brackets and write simply $v_\lambda^\star x \otimes y v_\lambda \mapsto v_\lambda^\star x y v_\lambda$ thanks to the cyclicity. The subspaces of different weights in M_λ^\star and M_λ are orthogonal. The module M_λ (equivalently, M_λ^\star) is irreducible if and only if this form is non-degenerate.

Recall that a vector in M_λ is called singular if it is annihilated by \mathfrak{n}^+ . Similarly, a vector in M_λ^\star is called singular if it is annihilated by \mathfrak{n}^- . Singular vectors generate submodules, where they carry the highest weights. For a subalgebra $\mathfrak{g}_{ij} \subset \mathfrak{g}$ we say that a vector in M_λ is \mathfrak{g}_{ij} -singular or \mathfrak{n}_{ij}^+ -singular if it is killed by \mathfrak{n}_{ij}^+ . Similarly, we say that a vector in M_λ^\star is \mathfrak{g}_{ij} -singular or \mathfrak{n}_{ij}^- -singular if it is killed by \mathfrak{n}_{ij}^- .

It is also convenient to extend the form to a cyclic pairing $M_\mu^\star \otimes M_\lambda \rightarrow \mathbb{C}$ by setting it nil for $\mu \neq \lambda$. Given a root subsystem $\Pi' \subset \Pi$, consider the corresponding semisimple Lie subalgebra $\mathfrak{g}' \subset \mathfrak{g}$. Suppose vectors $v_{\lambda'} \in M_{\lambda'}$ and $v_{\mu'}^\star \in M_{\mu'}^\star$ are \mathfrak{g}' -singular and consider the $U_q(\mathfrak{g}')$ -submodules $M_{\lambda'}' \subset M_{\lambda'}$ and $M_{\mu'}^{\star'} \subset M_{\mu'}^\star$ generated by $v_{\lambda'}$ and $v_{\mu'}^{\star'}$.

Proposition 4.1. *The restriction of the $U_q(\mathfrak{g})$ -cyclic form $M_\lambda^\star \otimes M_\lambda \rightarrow \mathbb{C}$ to $M_{\mu'}^{\star'} \otimes M_{\lambda'}'$ is proportional to the $U_q(\mathfrak{g}')$ -cyclic form $M_{\mu'}^{\star'} \otimes M_{\lambda'}' \rightarrow \mathbb{C}$.*

Proof. The restriction of the form to $M_{\mu'}^{\star'} \otimes M_{\lambda'}'$ is cyclic with respect to $U_q(\mathfrak{g}')$. A cyclic bilinear form between right and left Verma modules is unique up to an overall factor. \square

5 Diagonalization of the Shapovalov form

Let $\mathfrak{T} = \mathbb{Z}_+^{\frac{n(n+1)}{2}}$ designate the set of triangular arrays $\mathbf{l} = (l_{ij})_{1 \leq i \leq j \leq n}$ with non-negative integer entries l_{ij} . For every $\mathbf{l} \in \mathfrak{T}$ and $k \in [1, n]$ we denote by $\mathbf{l}_k \in \mathbb{Z}_+^{n-k+1}$ its k -th row $(l_{kj})_{k \leq j \leq n}$. Define

$$\begin{aligned} f(\mathbf{l}_k) &= f_{kn}^{l_{kn}} \dots f_{kk}^{l_{kk}} \in U(\mathfrak{b}^-), & f(\mathbf{l}) &= f(\mathbf{l}_n) \dots f(\mathbf{l}_1), \\ e(\mathbf{l}_k) &= e_{kk}^{l_{kk}} \dots e_{kn}^{l_{kn}} \in U(\mathfrak{b}^+), & e(\mathbf{l}) &= e(\mathbf{l}_1) \dots e(\mathbf{l}_n). \end{aligned}$$

The set $\{f(\mathbf{l}), e(\mathbf{l})\}_{\mathbf{l} \in \mathfrak{T}} \subset U_q(\mathfrak{g})$ is a PBW basis over $U_q(\mathfrak{h})$. Similarly we define $\hat{f}(\mathbf{l})$ and $\hat{e}(\mathbf{l})$ using the dynamical root vectors in place of standard. We call $\{\hat{f}(\mathbf{l}), \hat{e}(\mathbf{l})\}_{\mathbf{l} \in \mathfrak{T}}$ dynamical PBW system. In what follows, we study the set of vectors

$$\hat{f}(\mathbf{l})v_\lambda \in M_\lambda, \quad v_\lambda^* \hat{e}(\mathbf{l}) \in M_\lambda^*, \quad \mathbf{l} \in \mathfrak{T}. \quad (5.2)$$

We prove that, upon a normalization, they form dual bases in generic M_λ and M_λ^* with respect to the cyclic pairing. With respect to the contravariant form on generic M_λ , the system $\{\hat{f}(\mathbf{l})v_\lambda\}_{\mathbf{l} \in \mathfrak{T}}$ is an orthogonal basis.

Note that the ordering of the dynamical root vectors is the same lexicographic ordering of the standard root vectors set up in Section 3. We call it *normal*. We have to consider different row-wise orderings as well. Let $\sigma = (\sigma_n, \dots, \sigma_1) \in S_n \times \dots \times S_1$ be an n -tuple of permutations. Define $\hat{e}_\sigma(\mathbf{l}_k) = \sigma_k(\hat{e}(\mathbf{l}_k))$ to be the result of permutation σ_k applied to the simple factors of $\hat{e}(\mathbf{l}_k)$ and put $\hat{e}_\sigma(\mathbf{l}) = \hat{e}_{\sigma_1}(\mathbf{l}_1) \dots \hat{e}_{\sigma_n}(\mathbf{l}_n)$. We prove in Section 7 that $\hat{e}_\sigma(\mathbf{l})$ is independent of σ but we have to distinguish between different orderings until then. We will suppress the subscript σ and understand by $\hat{e}(\mathbf{l})$ a monomial with arbitrary although fixed ordering. This convention stays in effect until the end of the section. In the subsequent sections, we use only two orderings: the normal and an alternative, for which we fix a special notation.

The basis of positive (negative) root vectors allows us to identify the factorspaces $\mathfrak{n}_{ij}^\pm / \mathfrak{n}_{kj}^\pm$ with the linear complements $\mathfrak{n}_{ij}^\pm \ominus \mathfrak{n}_{kj}^\pm \subset \mathfrak{n}_{ij}^\pm$, for all $i, j, k \in [1, n]$ such that $i \leq k \leq j$. By $U_q(\mathfrak{n}_{ij}^\pm / \mathfrak{n}_{kj}^\pm)$ we denote the subalgebras in $U_q(\mathfrak{g})$ generated by $\mathfrak{n}_{ij}^\pm / \mathfrak{n}_{kj}^\pm$.

Similarly we define $U_q(\mathfrak{h})$ -submodules $\hat{\mathfrak{n}}_{ij}^+ = \text{Span}\{\hat{e}_{lk}\}_{i \leq l \leq k \leq j}$, $\hat{\mathfrak{n}}_{ij}^- = \text{Span}\{\hat{f}_{lk}\}_{i \leq l \leq k \leq j}$ and $\hat{\mathfrak{n}}_{ij}^\pm / \hat{\mathfrak{n}}_{kj}^\pm = \hat{\mathfrak{n}}_{ij}^\pm \ominus \hat{\mathfrak{n}}_{kj}^\pm \subset \hat{\mathfrak{n}}_{ij}^\pm$. By $U_q(\hat{\mathfrak{n}}_{ij}^\pm) \subset U_q(\mathfrak{b}_{ij}^\pm)$ we denote the subalgebras generated by $\hat{\mathfrak{n}}_{ij}^\pm$ and by $U_q(\hat{\mathfrak{n}}_{ij}^\pm / \hat{\mathfrak{n}}_{kj}^\pm)$ the subalgebras generated by $\hat{\mathfrak{n}}_{ij}^\pm / \hat{\mathfrak{n}}_{kj}^\pm$. Clearly $U_q(\hat{\mathfrak{n}}_{ij}^-)v_\mu \subset U_q(\mathfrak{n}_{ij}^-)v_\mu$ and $v_\mu^* U_q(\hat{\mathfrak{n}}_{ij}^+) \subset v_\mu^* U_q(\mathfrak{n}_{ij}^+)$ for all weight vectors v_μ, v_μ^* . The monomial structure of $f(\mathbf{l})v_\lambda$ is compatible with the factorization

$$U_q(\hat{\mathfrak{n}}_{1n}^-)v_\lambda = U_q(\hat{\mathfrak{n}}_{kn}^-)U_q(\hat{\mathfrak{n}}_{1n}^- / \hat{\mathfrak{n}}_{kn}^-)v_\lambda = U_q(\hat{\mathfrak{n}}_{nn}^-)U_q(\hat{\mathfrak{n}}_{n-1n}^- / \hat{\mathfrak{n}}_{nn}^-) \dots U_q(\hat{\mathfrak{n}}_{1n}^- / \hat{\mathfrak{n}}_{2n}^-)v_\lambda.$$

Similarly, the vector $v_\lambda^* \hat{e}(\mathbf{l})$ is factorized in accordance with

$$v_\lambda^* U_q(\hat{\mathfrak{n}}_{1n}^+) = v_\lambda^* U_q(\hat{\mathfrak{n}}_{1n}^+ / \hat{\mathfrak{n}}_{kn}^+)U_q(\hat{\mathfrak{n}}_{kn}^+) = v_\lambda^* U_q(\hat{\mathfrak{n}}_{1n}^+ / \hat{\mathfrak{n}}_{2n}^+) \dots U_q(\hat{\mathfrak{n}}_{n-1n}^+ / \hat{\mathfrak{n}}_{nn}^+)U_q(\hat{\mathfrak{n}}_{nn}^+).$$

We shall see in Section 7 that the algebras $U_q(\hat{\mathfrak{n}}_{in}^\pm / \hat{\mathfrak{n}}_{i+1n}^\pm)$ are commutative.

Lemma 5.1. *Suppose that $1 \leq k \leq n$. Then all vectors from $U_q(\hat{\mathfrak{n}}_{1n}^- / \hat{\mathfrak{n}}_{kn}^-)v_\lambda$ and $v_\lambda^* U_q(\hat{\mathfrak{n}}_{1n}^+ / \hat{\mathfrak{n}}_{kn}^+)$ are \mathfrak{g}_{kn} -singular.*

Proof. An immediate consequence of Proposition 3.1. \square

Fix $\mathbf{l} \in \mathfrak{T}$ and define a sequence of weights $(\lambda_{\mathbf{l},i})_{i=0}^n \subset \mathfrak{h}^*$ by

$$\lambda_{\mathbf{l},0} = \lambda, \quad \lambda_{\mathbf{l},i} = \lambda_{\mathbf{l},i-1} - \sum_{i \leq j \leq k \leq n} l_{ik} \alpha_j, \quad i = 1, \dots, n. \quad (5.3)$$

These are the weights of the vectors $\hat{f}(\mathbf{l}_i) \dots \hat{f}(\mathbf{l}_1) v_\lambda$. Note that the difference $\lambda_{\mathbf{l},i} - \lambda_{\mathbf{l},i-1}$ depends only on \mathbf{l}_i and not on λ . Define vectors

$$v_{\lambda_{\mathbf{k},0}}^* = v_\lambda^*, \quad v_{\lambda_{\mathbf{l},0}} = v_\lambda, \quad v_{\lambda_{\mathbf{k},i}}^* = v_\lambda^* \hat{e}(\mathbf{k}_n) \dots \hat{e}(\mathbf{k}_i) \in M_\lambda^*, \quad v_{\lambda_{\mathbf{l},i}} = \hat{f}(\mathbf{l}_i) \dots \hat{f}(\mathbf{l}_1) v_\lambda M_\lambda, \quad (5.4)$$

$i \in [1, n]$, of weights $\lambda_{\mathbf{l},i}$ (mind the right action of $U_q(\mathfrak{h})$ on M_λ^*).

Proposition 5.2. *For all $\mathbf{k}, \mathbf{l} \in \mathfrak{T}$, the matrix coefficient $v_\lambda^* \hat{e}(\mathbf{k}) \hat{f}(\mathbf{l}) v_\lambda$ is nil unless $\mathbf{k} = \mathbf{l}$.*

Proof. Due to Lemma 5.1, for each i the vector $v_{\lambda_{\mathbf{l},i}} \in U_q(\hat{\mathfrak{n}}_{1n}^- / \hat{\mathfrak{n}}_{i+1n}^-) v_\lambda$ is \mathfrak{g}_{i+1n} -singular. Let $M_{\lambda_{\mathbf{l},i}} = U_q(\hat{\mathfrak{n}}_{i+1n}^-) v_{\lambda_{\mathbf{l},i}}$ denote the $U_q(\mathfrak{g}_{i+1n})$ -Verma submodule in M_λ generated by $v_{\lambda_{\mathbf{l},i}}$. In the similar way we define the $U_q(\mathfrak{g}_{i+1n})$ -Verma submodule $M_{\lambda_{\mathbf{k},i}}^* = v_{\lambda_{\mathbf{k},i}}^* U_q(\hat{\mathfrak{n}}_{i+1n}^+)$ in M_λ^* generated by a \mathfrak{g}_{i+1n} -singular vector $v_{\lambda_{\mathbf{k},i}}^* \in v_\lambda^* U_q(\hat{\mathfrak{n}}_{1n}^+ / \hat{\mathfrak{n}}_{i+1n}^+)$.

By construction, $\lambda_{\mathbf{k},0} = \lambda_{\mathbf{l},0} = \lambda$. Suppose that we have proved the equality $\lambda_{\mathbf{k},i-1} = \lambda_{\mathbf{l},i-1}$ for some $i \in [1, n]$. Then $v_\lambda^* \hat{e}(\mathbf{k}) \hat{f}(\mathbf{l}) v_\lambda$ can be presented as the matrix coefficient $v_{\lambda_{\mathbf{l},i-1}}^* \hat{e}(\mathbf{k}_i) \dots \hat{e}(\mathbf{k}_n) \hat{f}(\mathbf{l}_n) \dots \hat{f}(\mathbf{l}_i) v_{\lambda_{\mathbf{l},i-1}}$ of a cyclic paring between the $U_q(\mathfrak{g}_{i+1n})$ -modules $M_{\lambda_{\mathbf{k},i}}^*$ and $M_{\lambda_{\mathbf{l},i}}$. It is zero unless $\lambda_{\mathbf{k},i} = \lambda_{\mathbf{l},i}$. This is true for all $i \in [0, n]$, by induction on i .

The equalities $\lambda_{\mathbf{k},i} - \lambda_{\mathbf{k},i-1} = \lambda_{\mathbf{l},i} - \lambda_{\mathbf{l},i-1}$ for $i \in [1, n]$ translate to a triangular system of equations on the differences $k_{is} - l_{is}$: namely, $\sum_{s=j}^n (k_{is} - l_{is}) = 0$ for all $j = i, \dots, n$. It is immediate that $\mathbf{k}_i = \mathbf{l}_i$ for all $i \in [1, n]$ and therefore $\mathbf{k} = \mathbf{l}$. \square

It follows that $U_q(\hat{\mathfrak{n}}_{kn}^-) v_\lambda$ is orthogonal to $v_\lambda^* U_q(\hat{\mathfrak{n}}_{in}^+ / \hat{\mathfrak{n}}_{kn}^+)$ and $U_q(\hat{\mathfrak{n}}_{in}^- / \hat{\mathfrak{n}}_{kn}^-) v_\lambda$ is orthogonal to $v_\lambda^* U_q(\hat{\mathfrak{n}}_{kn}^+)$ for all $i, k \in [1, n]$, $i < k$. Calculation of (5.2) boils down to calculation of the matrix coefficients

$$v_\mu^* \hat{e}(\mathbf{l}_k) \hat{f}(\mathbf{l}_k) v_\mu, \quad 1 \leq k \leq n,$$

where $v_\mu \in M_\lambda$ and $v_\mu^* \in M_\lambda^*$ are \mathfrak{g}_{kn} -singular vectors. This is done in the following section.

6 The matrix coefficients

Given a weight $\mu \in \mathfrak{h}^*$, we put $\mu_i = (\mu, \alpha_i)$ and $\mu_{ij} = \mu_i + \dots + \mu_j + j - i$, assuming $i \leq j \leq n$. We adopt the convention that products $\prod_{i=a}^b$ are not implemented (formally set to 1) once

$a > b$. For every $\mathbf{l} \in \mathfrak{T}$ and every $k \in [1, n]$ we define

$$A_{\mathbf{l},k}(\mu) = \prod_{k+1 \leq s \leq r \leq n} \prod_{i=0}^{l_r-1} [\mu_{sr} - i + l_{s-1} + 1]_q, \quad \mathbf{l}_k = (l_n, \dots, l_k).$$

According to this definition, $A_{\mathbf{l},k}(\mu)$ actually depends on the k -th row $\mathbf{l}_k \in \mathbb{Z}_+^{n-k+1}$ of \mathbf{l} .

Lemma 6.1. *The matrix coefficient $v_\lambda^* \hat{e}(\mathbf{l}_1) \hat{f}(\mathbf{l}_1) v_\lambda$ is equal to $A_{\mathbf{l},1}(\lambda) v_\lambda^* \hat{e}(\mathbf{l}_1) f(\mathbf{l}_1) v_\lambda$.*

Proof. The element $\hat{f}(\mathbf{l}_1)$ is a monomial in the dynamical root vectors \hat{f}_{1m} , where m ranges from 1 to n . The element \hat{f}_{1m} is a sum of monomials in f_1, \dots, f_m with coefficients from the Cartan subalgebra. Let us prove that only $f_1 \dots f_m$ survives in each copy of \hat{f}_{1m} . The other monomials, which are obtained by a permutation of the simple root vectors f_i , vanish in the matrix coefficient. Suppose we have replaced all \hat{f}_{1m} with $f_1 \dots f_m \prod_{i=2}^m [h_{im} + 1]_q$ on the left of some factor \hat{f}_{1k} and denote the result by ψ , i.e., $f(\mathbf{l}_1) v_\lambda = \psi \hat{f}_{1k} \dots v_\lambda$. The element ψ is a product of the monomials $f_1 \dots f_m$ with $m \geq k$, and $f_1 \dots f_m \in \mathfrak{n}_{2m}^- U$ by Lemma 2.2. This implies $\psi \mathfrak{n}_{2k}^- \subset \mathfrak{n}_{2n}^- U$. Every monomial $\phi = f_{\sigma(1)} \dots f_{\sigma(k)}$ entering \hat{f}_{1k} with $\sigma \neq \text{id}$ belongs to $\mathfrak{n}_{2k}^- U$ by Lemma 2.3. Therefore, the vector $v_\lambda^* \hat{e}_{\mathbf{l}_1} \psi \phi \in v_\lambda^* \hat{e}_{\mathbf{l}_1} \mathfrak{n}_{2n}^- U$ is nil.

By this reasoning, we can consecutively replace each \hat{f}_{1m} with $f_1 \dots f_m \prod_{i=2}^m [h_{im} + 1]_q$ factor by factor from left to right. The Cartan coefficients produce scalar multipliers, which gather to the overall factor $A_{\mathbf{l},1}(\lambda)$. Finally, we replace each $f_1 \dots f_m$ with f_{1m} by a similar reasoning moving in the opposite direction, from right to left. \square

Next we calculate the matrix coefficient $v_\lambda^* \hat{e}_{1n}^l f_{1n}^l v_\lambda$. For all $k, m \in [1, n]$ such that $k \leq m$ we define polynomial functions $C_{km}: \mathfrak{h}^* \rightarrow \mathbb{C}$ by

$$\lambda \mapsto C_{km}(\lambda) = \prod_{i=k}^m [\lambda_{im}]_q. \quad (6.5)$$

Lemma 6.2. *The matrix coefficient $v_\lambda^* \hat{e}_{1n} f_{1n} v_\lambda$ is equal to $C_{1n}(\lambda)$.*

Proof. We do induction on n . The statement for $n = 1$ immediately follows from the defining relations. Suppose that $n > 1$ and present f_{1n} as $f_1 f_{2n} - q f_{2n} f_1$. Observe that $v_\lambda^* \hat{e}_{1n} f_{2n} f_1 v_\lambda$ vanishes since the vector $v_\lambda^* \hat{e}_{1n}$ is \mathfrak{n}_{2n}^- -singular by Proposition 5.1. Now plug $\hat{e}_{1n} = [h_{2n} + 1]_q \hat{e}_{2n} e_1 - [h_{2n}]_q e_1 \hat{e}_{2n}$ in $v_\lambda^* \hat{e}_{1n} f_{1n} v_\lambda = v_\lambda^* \hat{e}_{1n} f_1 f_{2n} v_\lambda$ and push f_1 to the left. Observe that f_1 commutes with \hat{e}_{2n} . The commutators of f_1 with the Cartan factors can be also neglected, as f_1 kills v_λ^* . We get for $v_\lambda^* \hat{e}_{1n} f_{1n} v_\lambda$ the expression

$$v_\lambda^* ([h_{2n} + 1]_q \hat{e}_{2n} [h_1]_q - [h_{2n}]_q [h_1]_q \hat{e}_{2n}) f_{2n} v_\lambda = [\lambda_{1n}]_q v_\lambda^* \hat{e}_{2n} f_{2n} v_\lambda,$$

since $([h_{2n} + 1]_q[h_1 + 1]_q - [h_{2n}]_q[h_1]_q = [h_1 + h_{2n} + 1]_q = [h_{1n}]_q$. Therefore, $v_\lambda^* \hat{e}_{1n} f_{1n} v_\lambda = [\lambda_{1n}]_q v_\lambda^* \hat{e}_{2n} f_{2n} v_\lambda = C_{1n}(\lambda)$, by the straightforward induction. \square

In the matrix coefficient $v_\lambda^* \hat{e}_{1n}^{l-1} \hat{e}_{1n} f_{1n}^l v_\lambda$, present the rightmost copy of \hat{e}_{1n} as a sum of Chevalley monomials $e_{\sigma(1)} \dots e_{\sigma(n)}$, $\sigma \in S_n$, with coefficients from $U_q(\mathfrak{h})$. By Lemma 2.1, the generators e_i commute with f_{1n} for all $i \in [2, n-1]$. Therefore, non-zero contributions to the matrix coefficient are made only by the monomials

$$\phi_1 = e_1 \dots e_n, \quad \phi_i = e_i \dots e_n e_{i-1} \dots e_1, \quad \phi_n = e_n \dots e_1,$$

where $i \in (1, n)$. Let us calculate $\phi_i f_{1n}^l v_\lambda$. We do it modulo $\mathfrak{n}_{2n}^- U v_\lambda$, which disappears when paired with $v_\lambda^* \hat{e}_{1n}^{l-1}$.

For every $i = 1, \dots, n$ and all $l \in \mathbb{N}$, define functions $D_{i,l}: \mathfrak{h}^* \rightarrow \mathbb{C}$ by

$$\begin{aligned} D_{1,l}(\lambda) &= q^{-l+1} [l]_q (-q)^{n-1} q^{\lambda_2 + \dots + \lambda_n} [\lambda_1]_q, \\ D_{i,l}(\lambda) &= q^{-l+1} [l]_q (-q)^{n-i} q^{-\lambda_1 - \dots - \lambda_{i-1} + \lambda_{i+1} + \dots + \lambda_n} [\lambda_i]_q, \quad i \in [2, n-1], \\ D_{n,l}(\lambda) &= q^{-l+1} [l]_q q^{l-1} q^{-\lambda_1 - \dots - \lambda_{n-1}} [\lambda_n - l + 1]_q. \end{aligned}$$

Lemma 6.3. *The action of the monomials ϕ_i , $i \in [1, n]$, on the vectors $f_{1n}^l v_\lambda$, $l \in \mathbb{N}$, is given by $\phi_i f_{1n}^l v_\lambda = D_{i,l}(\lambda) f_{1n}^{l-1} v_\lambda \pmod{\mathfrak{n}_{2n}^- U v_\lambda}$.*

Proof. Assuming $i \in [2, n]$, present ϕ_i as $\phi'_i e_1$, where $\phi'_i \in U_q(\mathfrak{n}_{2n}^+)$. Observe that the relation $f_{2n} f_{1n} = q f_{1n} f_{2n}$ easily follows from Lemma 2.1. Along with the relation $[e_1, f_{1n}] = f_{2n} q^{-h_1}$ from the same lemma, this yields

$$\phi_i f_{1n}^l v_\lambda = \phi'_i f_{2n} q^{-h_1} f_{1n}^{l-1} v_\lambda + \phi'_i f_{1n} f_{2n} q^{-h_1} f_{1n}^{l-2} v_\lambda + \dots = [l]_q q^{-\lambda_1} \phi'_i f_{2n} f_{1n}^{l-1} v_\lambda.$$

Present ϕ'_i as $e_{s_1^i} \dots e_{s_{n-1}^i}$ and write

$$\phi'_i f_{2n} f_{1n}^{l-1} v_\lambda = e_{s_1^i} \dots e_{s_{n-1}^i} f_{2n} f_{1n}^{l-1} v_\lambda = [e_{s_1^i}, \dots, [e_{s_{n-1}^i}, (f_{2n})(f_{1n}^{l-1})] \dots] v_\lambda.$$

Applying the Leibnitz rule to these commutators, we can ignore f_{1n}^{l-1} :

$$e_{s_1^i} \dots e_{s_{n-1}^i} f_{2n} f_{1n}^{l-1} v_\lambda = [e_{s_1^i}, \dots, [e_{s_{n-1}^i}, f_{2n}] \dots] f_{1n}^{l-1} v_\lambda + \dots$$

The omitted terms contain residual vectors coming from f_{2n} . They lie in $\mathfrak{n}_{2n}^- U$ and vanish in the matrix coefficient. Modulo $\mathfrak{n}_{2n}^- U$, Lemma 2.1 yields

$$\phi_i f_{1n}^l v_\lambda = q^{-l'} [l]_q q^{\delta_{in} l'} (-q)^{n-i} q^{-\lambda_1 - \dots - \lambda_{i-1} + \lambda_{i+1} + \dots + \lambda_n} [\lambda_i - \delta_{in} l']_q f_{1n}^{l-1} v_\lambda, \quad i \in [2, n],$$

where $l' = l - 1$. This proves the statement for ϕ_i , $i \in [2, n]$.

Consider the remaining case of ϕ_1 . Using the relation $[e_n, f_{1n}] = -qf_{1n-1}q^{h_n}$ and the relation $f_{1n-1}f_{1n} = q^{-1}f_{1n}f_{1n-1}$ from Lemma 2.1, we get

$$\phi_1 f_{1n}^l v_\lambda = -q[l]_q q^{\lambda_n} e_1 \dots e_{n-1} f_{1n-1} f_{1n}^{l-1} v_\lambda = (-q)^{n-1} [l]_q q^{\lambda_2 + \dots + \lambda_n} e_1 f_{1n}^{l-1} v_\lambda.$$

We have used $[e_i, f_{1n}] = 0$ for $i \in [2, n-1]$ in this calculation. Further,

$$e_1 f_{1n}^{l-1} v_\lambda = [\lambda_1 - l']_q f_{1n}^{l-1} v_\lambda + [l']_q q^{-\lambda_1} f_1 f_{2n} f_{1n}^{l-2} v_\lambda.$$

We replace the product $f_1 f_{2n}$ with f_{1n} , since the calculation is done modulo $\mathfrak{n}_{2n}^- U$. Thus,

$$\phi_1 f_{1n}^l v_\lambda = [l]_q (-q)^{n-1} q^{\lambda_n + \dots + \lambda_2} \left([\lambda_1 - l']_q + [l']_q q^{-\lambda_1} \right) f_{1n}^{l-1} v_\lambda \mod \mathfrak{n}_{2n}^- U.$$

Notice that the factor in the brackets is equal to $[\lambda_1 - l']_q + [l']_q q^{-\lambda_1} = q^{-l'} [\lambda_1]_q$. This completes the proof. \square

The coefficients $D_{i,l}(\lambda)$ satisfy the reduction formulas

$$D_{1,l}(\lambda) - q^{-l'} [l]_q (-1)^{n-1} q^{\lambda_{1n} - l'} [l']_q = q^{-l'} [l]_q D_{1,1}(\lambda - l' \alpha_{1n}), \quad (6.6)$$

$$D_{i,l}(\lambda) = q^{-l'} [l]_q D_{i,1}(\lambda - l' \alpha_{1n}), \quad i \in [2, n], \quad (6.7)$$

which readily follow from their definition. As above, $l' = l - 1$.

Lemma 6.4. *For all $l \in \mathbb{N}$,*

$$v_\lambda^* \hat{e}_{1n}^l f_{1n}^l v_\lambda = [l_q]! \prod_{i=0}^{l-1} [\lambda_{1n} - i]_q [\lambda_{2n} - i]_q \dots [\lambda_{nn} - i]_q. \quad (6.8)$$

Proof. Let us calculate the vector $\hat{e}_{1n} f_{1n}^l v_\lambda$ modulo $\mathfrak{n}_{2n}^- U v_\lambda$. Consider the presentation $\hat{e}_{1n} = \sum_{i=1}^n a_i(h) \phi_i + \dots$ with suppressed Chevalley monomials from $U \mathfrak{n}_{2n-1}^+$. They make zero contribution to the vector $\hat{e}_{1n} f_{1n}^l v_\lambda$, because \mathfrak{n}_{2n-1}^+ commutes with f_{1n} and kills $f_{1n}^l v_\lambda$, by Lemma 2.1. We need the explicit expression only for $a_1(h) = (-1)^{n-1} \prod_{i=2}^n [h_{in}]_q$, which is readily found from the definition of \hat{e}_{1n} . We replace \hat{e}_{1n} with its specialization at the weight $\lambda - l' \alpha_{1n}$ and write

$$\hat{e}_{1n} f_{1n}^l v_\lambda = \sum_{i=1}^n a_i(\lambda - l' \alpha_{1n}) D_{i,l}(\lambda) f_{1n}^{l-1} v_\lambda \mod \mathfrak{n}_{2n}^- U v_\lambda. \quad (6.9)$$

Observe that $\sum_{i=1}^n a_i(\mu) D_{i,1}(\mu) = C_{1n}(\mu)$ for all $\mu \in \mathfrak{h}^*$. For higher l , the coefficient $\sum_{i=1}^n a_i(\lambda - l'\alpha_{1n}) D_{i,l}(\lambda)$ before $f_{1n}^l v_\lambda$ in (6.9) is found to be

$$\begin{aligned} & \sum_{i=2}^n a_i(\lambda - l'\alpha_{1n}) D_{i,l}(\lambda) + a_1(\lambda - l'\alpha_{1n}) D_{1,l}(\lambda) \\ &= q^{-l'} [l]_q \sum_{i=2}^n a_i(\lambda - l'\alpha_{1n}) D_{i,1}(\lambda - l'\alpha_{1n}) + a_1(\lambda - l'\alpha_{1n}) D_{1,1}(\lambda) \\ &= q^{-l'} [l]_q C_{1n}(\lambda - l'\alpha_{1n}) + a_1(\lambda - l'\alpha_{1n}) (D_{1,l}(\lambda) - q^{-l'} [l]_q D_{1,1}(\lambda - l'\alpha_{1n})). \end{aligned}$$

We have used the reduction formulas (6.7) in the second equality. Plug in here the expressions

$$C_{1n}(\lambda - l'\alpha_{1n}) = [\lambda_{1n} - 2l']_q \prod_{k=2}^n [\lambda_{kn} - l']_q, \quad a_1(\lambda - l'\alpha_{1n}) = (-1)^{n-1} \prod_{k=2}^n [\lambda_{kn} - l']_q,$$

and the expression for the difference $D_{1,l}(\lambda) - q^{-l'} [l]_q D_{1,1}(\lambda - l'\alpha_{1n})$ from (6.6). This gives the coefficient before $f_{1n}^l v_\lambda$ in (6.9). It is divisible by $q^{-l'} [l]_q \prod_{k=2}^n [\lambda_{kn} - l']_q$, which can be factored out. The remaining factor is

$$[\lambda_{1n} - 2l']_q + [l']_q q^{\lambda_{1n} - l'} = \frac{q^{\lambda_{1n} - 2l'} - q^{-\lambda_{1n} + 2l'} + (q^{l'} - q^{-l'}) q^{\lambda_{1n} - l'}}{q - q^{-1}} = q^{l'} [\lambda_{1n} - l']_q.$$

Combining this with the multiplier $q^{-l'} [l]_q \prod_{k=2}^n [\lambda_{kn} - l']_q$ we obtain the recurrent formula $\langle v_\lambda^* \hat{e}_{1n}^l f_{1n}^l v_\lambda \rangle = [l]_q \prod_{k=1}^n [\lambda_{kn} - l']_q \langle v_\lambda^* \hat{e}_{1n}^{l-1} f_{1n}^{l-1} v_\lambda \rangle$. Induction on l completes the proof. \square

From now on we understand by $\hat{e}(\mathbf{l})$ the *normally* ordered PBW monomial. To proceed with the calculation of matrix coefficients of the cyclic Shapovalov pairing, we fix another ordering on the positive dynamical root vectors: we define

$$\check{e}(\mathbf{l}_k) = \hat{e}_{kn}^{l_{kn}} \dots \hat{e}_{kk}^{l_{kk}}, \quad \check{e}(\mathbf{l}) = \check{e}(\mathbf{l}_1) \dots \check{e}(\mathbf{l}_n).$$

In the last section we demonstrate that $\check{e}(\mathbf{l}) = \hat{e}(\mathbf{l})$, but the proof of this nontrivial fact is indirect and based on the knowledge of the matrix coefficients $v_\lambda^* \check{e}(\mathbf{l}_k) \hat{f}(\mathbf{l}_k) v_\lambda$.

Lemma 6.5. *Put $\mathbf{l}_1 = (l_n, \dots, l_1) \in \mathbb{Z}_+^n$. Then $v_\lambda^* \check{e}(\mathbf{l}_1) f(\mathbf{l}_1) v_\lambda = \prod_{k=1}^n v_\lambda^* \hat{e}_{1k}^{l_k} f_{1k}^{l_k} v_\lambda$.*

Proof. The above factorization of the matrix coefficient is a consequence of the formula

$$v_\lambda^* \hat{e}_{1n}^{l_n} \dots \hat{e}_{1k}^{l_k} f_{1n}^{l_n} \dots f_{1k}^{l_k} v_\lambda = v_\lambda^* (\hat{e}_{1n}^{l_n} \dots \hat{e}_{1k+1}^{l_{k+1}}) (f_{1n}^{l_n} \dots f_{1k+1}^{l_{k+1}}) \hat{e}_{1k}^{l_k} f_{1k}^{l_k} v_\lambda,$$

which holds true for all $k \in [1, n]$. Let us prove it. Denote by ψ the product $f_{1n}^{l_n} \dots f_{1k+1}^{l_{k+1}}$. It is sufficient to show that \hat{e}_{1k} commutes with ψ modulo $\mathfrak{n}_{2n}^- U$, as \mathfrak{n}_{2n}^- annihilates $v_\lambda^* \hat{e}_{1n}^{l_n} \dots \hat{e}_{1k}^{l_k - i - 1}$.

Let ν denote the weight of this vector and let $\tilde{e}_{1k} \in U_q(\mathfrak{n}_{1k}^+)$ be the specialization of \hat{e}_{1n} at ν . It follows from Lemma 2.3 and Lemma 2.2 that $[\tilde{e}_{1k}, \psi] \in \mathfrak{n}_{2n}^- U$. Therefore, we can replace $\hat{e}_{1k}\psi$ with $\psi\tilde{e}_{1k} \pmod{\mathfrak{n}_{2n}^- U}$. Finally, observe that the Cartan coefficients of \hat{e}_{1k} are confined within $U_q(\mathfrak{h}_{2k})$ and consequently commute with ψ . Therefore, $\psi\tilde{e}_{1k}$ can be replaced with $\psi\hat{e}_{1k}$ modulo $\mathfrak{n}_{2n}^- U$.

To finish the proof, observe that $\hat{e}_{1k}^{l_k} f_{1k}^{l_k} v_\lambda = \langle v_\lambda^* \hat{e}_{1k}^{l_k}, f_{1k}^{l_k} v_\lambda \rangle \times v_\lambda$. Varying k from 1 to n we prove the factorization of $v_\lambda^* \check{e}(\mathbf{l}_1) f(\mathbf{l}_1) v_\lambda$. \square

So far in this section we dealt with the matrix coefficients $v_\lambda^* \check{e}(\mathbf{l}_1) f(\mathbf{l}_1) v_\lambda$, i.e. of the form $v_\lambda^* U_q(\hat{\mathfrak{n}}_{1n}^+ / \hat{\mathfrak{n}}_{2n}^+) U_q(\hat{\mathfrak{n}}_{1n}^- / \hat{\mathfrak{n}}_{2n}^-) v_\lambda$. Upon obvious modifications, these results hold true for $v_\mu^* \check{e}(\mathbf{l}_k) f(\mathbf{l}_k) v_\mu$, for any $k \in [1, n]$ and $v_\mu^* \in M_\mu^*$, $v_\mu \in M_\mu$ being \mathfrak{g}_{kn} -singular vectors.

Corollary 6.6. *Suppose that $v_\mu \in M_\lambda$ and $v_\mu^* \in M_\lambda^*$ are \mathfrak{g}_{kn} -singular vectors of weight μ . Then the matrix coefficient $v_\mu^* \check{e}(\mathbf{l}_k) \hat{f}(\mathbf{l}_k) v_\mu$ is equal to*

$$[l_k]_q! \dots [l_n]_q! \prod_{k \leq s \leq r \leq n} \prod_{i=0}^{l_r-1} [\mu_{sr} - i]_q \times \prod_{k+1 \leq s \leq r \leq n} \prod_{i=0}^{l_r-1} [\mu_{sr} - i + l_{s-1} + 1]_q v_\mu^* v_\mu, \quad (6.10)$$

where $l_r = l_{kr}$, $r = k, \dots, n$.

Proof. Replacement of $\hat{f}(\mathbf{l}_k)$ with $f(\mathbf{l}_k)$ yields a scalar multiplier $A_{\mathbf{l},k}(\mu)$, as explained by Lemma 6.1; hence the last product. Factorization of $v_\mu^* \check{e}(\mathbf{l}_k) f(\mathbf{l}_k) v_\mu$ is established by Lemma 6.5 and Lemma 6.4; hence the first product with the factorials. \square

Denote the matrix coefficients from Corollary 6.6 by $B_{\mathbf{l}_k}(\mu)$ and define

$$B_{\mathbf{l}}(\lambda) = B_{\mathbf{l}_1}(\lambda_{\mathbf{l},0}) \dots B_{\mathbf{l}_n}(\lambda_{\mathbf{l},n-1}),$$

where the weights $\lambda_{\mathbf{l},i}$ are introduced in (5.3).

Theorem 6.7. *The matrix coefficient $v_\lambda^* \check{e}(\mathbf{k}) \hat{f}(\mathbf{l}) v_\lambda$ is equal to $\delta_{\mathbf{k},\mathbf{l}} B_{\mathbf{l}}(\lambda)$.*

Proof. The Kronecker symbol is justified in Proposition 5.2. Further, let $v_{\lambda_{\mathbf{l},i}} \in M_\lambda$ and $v_{\lambda_{\mathbf{l},i}}^* \in M_\lambda^*$, $i \in [0, n]$, be the vectors defined in (5.4), where the positive PBW monomial is ordered as $\check{e}_{\mathbf{l}}$. Due to Lemma 5.1, the matrix coefficient $v_\lambda^* \check{e}(\mathbf{l}) \hat{f}(\mathbf{l}) v_\lambda$ factorizes to

$$v_{\lambda_{\mathbf{l},n-1}}^* \check{e}(\mathbf{l}_n) \hat{f}(\mathbf{l}_n) v_{\lambda_{\mathbf{l},n-1}} = B_{\mathbf{l}_n}(\lambda_{\mathbf{l},n-1}) v_{\lambda_{\mathbf{l},n-1}}^* v_{\lambda_{\mathbf{l},n-1}} = \dots = B_{\mathbf{l}_n}(\lambda_{\mathbf{l},n-1}) \dots B_{\mathbf{l}_1}(\lambda_{\mathbf{l},0}) v_{\lambda_{\mathbf{l},0}}^* v_{\lambda_{\mathbf{l},0}},$$

where $v_{\lambda_{\mathbf{l},0}}^* v_{\lambda_{\mathbf{l},0}} = v_\lambda^* v_\lambda = 1$. At every step $k \in [1, n]$ we apply Corollary 6.6 in order to calculate the matrix coefficient $v_{\lambda_{\mathbf{l},k-1}}^* \check{e}(\mathbf{l}_k) \hat{f}(\mathbf{l}_k) v_{\lambda_{\mathbf{l},k-1}}$ with $\check{e}(\mathbf{l}_k), \hat{f}(\mathbf{l}_k) \in U_q(\mathfrak{g}_{kn})$ and the \mathfrak{g}_{kn} -singular vectors $v_{\lambda_{\mathbf{l},k-1}}^*, v_{\lambda_{\mathbf{l},k-1}}$ of weight $\mu = \lambda_{\mathbf{l},k-1}$. \square

Corollary 6.8. *Suppose the weight λ is such that $B_{\mathbf{l}}(\lambda) \neq 0$ for all $\mathbf{l} \in \mathfrak{T}$. Then the system $\{\hat{f}(\mathbf{l})v_\lambda\}_{\mathbf{l} \in \mathfrak{T}}$ forms a basis in M_λ and $\{\frac{1}{B_{\mathbf{l}}(\lambda)}v_\lambda^* \check{e}(\mathbf{l})\}_{\mathbf{l} \in \mathfrak{T}}$ is its dual basis in M_λ^* . The formal sum $\sum_{\mathbf{l}} \frac{1}{B_{\mathbf{l}}(\lambda)} \hat{f}(\mathbf{l})v_\lambda \otimes v_\lambda^* \check{e}(\mathbf{l}) \in M_\lambda \otimes M_\lambda^*$ is the inverse of the cyclic Shapovalov pairing.*

Proof. Both systems $\{\hat{f}(\mathbf{l})v_\lambda\}$, $\{v_\lambda^* \check{e}(\mathbf{l})\}$ have the same number of vectors of a given weight as their standard PBW counterparts. Their Gram matrix with respect to the Shapovalov pairing is non-degenerate, provided all $B_{\mathbf{l}}(\lambda) \neq 0$. Therefore, for such λ , $\{\hat{f}(\mathbf{l})v_\lambda\}$ is a basis in M_λ and $\{\frac{1}{B_{\mathbf{l}}(\lambda)}v_\lambda^* \check{e}(\mathbf{l})\}$ is its dual in M_λ^* , according to Theorem 6.7. \square

The classical version of these results is straightforward. One should pass to the algebra $U_{\hbar}(\mathfrak{g})$ and take the zero fiber mod \hbar . This operation converts $[x]_q$ into x for any indeterminate x . The classical version of the dynamical root vectors and the formulas for the matrix coefficients are immediate.

Recall from [6, 7] that the Shapovalov form on M_λ is invertible if and only if $q^{2(\lambda+\rho, \alpha)} \notin q^{2\mathbb{N}}$ (respectively, $(\lambda, \alpha) + (\rho, \alpha) \notin \mathbb{N}$ for $U(\mathfrak{g})$) for all $\alpha \in R^+$. In our notation, this criterion translates to $q^{2\lambda_{ij}} \notin q^{2\mathbb{Z}^+}$ (respectively, $\lambda_{ij} \notin \mathbb{Z}_+$) for all i, j such that $i \leq j$. On the other hand, one can easily see that the set of zeros of $B_{\mathbf{l}}(\lambda)$, $\mathbf{l} \in \mathfrak{T}$, is larger although contained in the union $\cup_{\alpha \in R^+} \{\lambda | q^{2(\lambda, \alpha)} \in q^{2\mathbb{Z}}\}$ (in the union of integer hyperplanes $(\lambda, \alpha) \in \mathbb{Z}$ in the classical case). Therefore, the system $\hat{f}(\mathbf{l})v_\lambda$, $\mathbf{l} \in \mathfrak{T}$, fails to be a basis for special values of weights. We consider this effect in a more detail on the example of $\mathfrak{sl}(3)$ in the last section.

Example 6.9. Here is an example which will play a role in the next section. We need the explicit expression for the matrix coefficient $v_\lambda^* \hat{e}_{1m} \hat{e}_{1k} \hat{f}_{1m} \hat{f}_{1k} v_\lambda$, $k < m$, which is

$$\prod_{j=2}^k (\lambda_{jk} + 1) \prod_{j=2}^k [\lambda_{jm} + 1]_q [\lambda_{k+1m} + 2]_q \prod_{j=k+2}^m [\lambda_{jm} + 1]_q C_{1k}(\lambda) C_{1m}(\lambda), \quad (6.11)$$

according to the general formula. As usual, the products are present only if the lower bounds do not exceed the upper bounds. The products before $C_{1k}(\lambda) C_{1m}(\lambda)$ results from the transition $\hat{f}_{1k} \rightarrow f_{1k}$, $\hat{f}_{1m} \rightarrow f_{1m}$. The Cartan coefficients $[h_{im} + 1]_q$ from \hat{f}_{1m} commute with f_{1k} unless $i = k + 1$, while $[h_{k+1m}, f_{1k}] = f_{1k}$. This accounts for 2 in the corresponding factor.

7 Contravariant Shapovalov form

In this section we refine the obtained results and show that the dual bases in M_λ^* and M_λ give rise to an orthogonal basis for the contravariant form on M_λ . The key step is to prove that

the dynamical positive (negative) root vectors commute within each row. This facilitates the equalities $\check{e}(\mathbf{l}_k) = \hat{e}(\mathbf{l}_k)$ for all $k \in [1, n]$ and $\check{e}(\mathbf{l}) = \hat{e}(\mathbf{l})$ for all $\mathbf{l} \in \mathfrak{T}$.

We start with the following simple case, which will be the base for a further induction.

Lemma 7.1. *For all $m \in [1, n]$, one has $[f_1, \hat{f}_{1m}] = 0$ and $[e_1, \hat{e}_{1m}] = 0$.*

Proof. This is an immediate consequence of the Serre relation:

$$\begin{aligned} f_1 f_{1.m} &= f_1 (f_1 \hat{f}_{2.m} [h_{2.m} + 1]_q - \hat{f}_{2.m} f_1 [h_{2.m}]_q) \\ &= f_1 \hat{f}_{2.m} f_1 ([2]_q [h_{2.m} + 1]_q - [h_{2.m}]_q) - \hat{f}_{2.m} f_1^2 [h_{2.m} + 1]_q = f_{1.m} f_1, \end{aligned}$$

since the difference in the brackets is equal to $[h_{2.m} + 2]_q$. Applying ω to $[f_1, \hat{f}_{1m}] = 0$ gives $[e_1, \hat{e}_{1m}] = 0$. \square

One can directly check $[\hat{f}_{12}, \hat{f}_{1m}] = 0$ for all m via a more cumbersome calculation. We have not found a direct general proof, apart from the above simplest cases, and use a roundabout approach based on already obtained results. Namely, we will show that positive dynamical PBW system vanishes when paired with the element $[\hat{f}_{1k}, \hat{f}_{1m}]v_\lambda$ for all λ . Since it is a basis in M_λ^\star and the pairing is non-degenerate for generic λ , that will be sufficient to prove the equality $[\hat{f}_{1k}, \hat{f}_{1m}] = 0$.

Proposition 7.2. *For every $i \in [1, n)$, the algebra $U_q(\mathfrak{n}_{in}^\pm / \mathfrak{n}_{i+1n}^\pm)$ is commutative.*

Proof. It is sufficient to check only $U_q(\mathfrak{n}_{in}^- / \mathfrak{n}_{i+1n}^-)$, thanks to the Chevalley involution. This algebra is generated by \hat{f}_{ik} , $k = i, \dots, n$. To prove the equality $[\hat{f}_{ik}, \hat{f}_{im}] = 0$, we do induction on $k - i$, where k is assumed to be less than m .

The case $k - i = 0$ is already established by Lemma 7.1. For higher k and $m \geq 3$, let us prove that the vector $[\hat{f}_{ik}, \hat{f}_{im}]v_\lambda \in M_\lambda$ is annihilated by $v_\lambda^\star U_q(\hat{\mathfrak{n}}_{1n}^+)$ for all λ . It suffices to restrict to $v_\lambda^\star U_q(\hat{\mathfrak{n}}_{in}^+)$, because $[\hat{f}_{ik}, \hat{f}_{im}]v_\lambda \in U_q(\hat{\mathfrak{n}}_{in}^-)v_\lambda$. Therefore, we can assume $i = 1$. Since $v_\lambda^\star \hat{U}_q(\mathfrak{n}_{2n}^+)[\hat{f}_{1k}, \hat{f}_{1m}]v_\lambda = 0$, we can restrict to $v_\lambda^\star U_q(\hat{\mathfrak{n}}_{1n}^+ / \hat{\mathfrak{n}}_{2n}^+)$. By weight arguments, it is sufficient to calculate the matrix element $v_\lambda^\star \hat{e}_{1m} \hat{e}_{1k} \hat{f}_{1k} \hat{f}_{1m} v_\lambda$ and check it against $v_\lambda^\star \hat{e}_{1m} \hat{e}_{1k} \hat{f}_{1m} \hat{f}_{1k} v_\lambda$, which is given in Example 6.9.

Plugging the expression $\hat{f}_{1k} = (f_1 \hat{f}_{2k} [h_{2k} + 1]_q - \hat{f}_{2k} f_1 [h_{2k}]_q)$ in $v_\lambda^\star \hat{e}_{1m} \hat{e}_{1k} \hat{e}_{1k} \hat{f}_{1m} v_\lambda$ we get $[\lambda_{2k} + 1]_q v_\lambda^\star \hat{e}_{1m} \hat{e}_{1k} f_1 \hat{f}_{2k} \hat{f}_{1m} v_\lambda$, since the second term makes zero contribution. Developing \hat{e}_{1k} in the similar way we continue to

$$v_\lambda^\star \hat{e}_{1m} \hat{e}_{1k} \hat{e}_{1k} \hat{f}_{1m} v_\lambda = [\lambda_{2k} + 1]_q v_\lambda^\star \hat{e}_{1m} ([h_{2k} + 1]_q \hat{e}_{2k} e_1 - [h_{2k}]_q e_1 \hat{e}_{2k}) f_1 \hat{f}_{2k} \hat{f}_{1m} v_\lambda. \quad (7.12)$$

Observe that h_{2k} commutes with \hat{e}_{1m} . The second term gives $-[\lambda_{2k} + 1]_q[\lambda_{2k}]_q$ times

$$v_\lambda^* \hat{e}_{1m} e_1 f_1 \hat{e}_{2k} \hat{f}_{2k} \hat{f}_{1m} v_\lambda = C_{2k}(\lambda) \prod_{j=3}^k [\lambda_{jk} + 1]_q v_\lambda^* \hat{e}_{1m} e_1 \hat{f}_{1m} f_1 v_\lambda.$$

In accordance with our convention, the product is replaced by 1 if $k = 2$. We have used the fact that $\hat{f}_{1m} v_\lambda$ is \mathfrak{n}_{2k}^+ -singular and $\hat{e}_{2k} \hat{f}_{2k} \hat{f}_{1m} v_\lambda = \langle v_\lambda^* \hat{e}_{2k}, \hat{f}_{2k} \rangle \hat{f}_{1m} v_\lambda$. Also, we have applied Lemma 7.1. The matrix coefficient in the right-hand side is standard, and can be specialized from the general formula (6.11). The contribution of the second term in (7.12) is

$$-[\lambda_{2k}]_q [\lambda_1]_q [\lambda_{2m} + 2]_q \prod_{j=2}^k [\lambda_{jk} + 1]_q \prod_{j=3}^m [\lambda_{jm} + 1]_q C_{2k}(\lambda) C_{1m}(\lambda). \quad (7.13)$$

Here we have used $C_{2k}(\lambda - \alpha_{1m}) = C_{2k}(\lambda)$, which is true for $k < m$.

The first term in (7.12) gives $[\lambda_{2k} + 1]_q^2$ times

$$v_\lambda^* \hat{e}_{1m} \hat{e}_{2k} e_1 f_1 \hat{f}_{2k} \hat{f}_{1m} v_\lambda = [\lambda_1]_q v_\lambda^* \hat{e}_{1m} \hat{e}_{2k} \hat{f}_{2k} \hat{f}_{1m} v_\lambda + v_\lambda^* \hat{e}_{1m} \hat{e}_{2k} f_1 e_1 \hat{f}_{2k} \hat{f}_{1m} v_\lambda.$$

The first matrix coefficient is standard and can be extracted from Theorem 6.7. The total contribution of this term to (7.12) is

$$[\lambda_{2k} + 1]_q [\lambda_1]_q \prod_{j=2}^k [\lambda_{jk} + 1]_q \prod_{j=2}^m [\lambda_{jm} + 1]_q C_{2k}(\lambda) C_{1m}(\lambda), \quad (7.14)$$

since $C_{2k}(\lambda - \alpha_{1m}) = C_{2k}(\lambda)$. Let us compute the matrix coefficient $v_\lambda^* \hat{e}_{1m} \hat{e}_{2k} f_1 e_1 \hat{f}_{2k} \hat{f}_{1m} v_\lambda = v_\lambda^* \hat{e}_{1m} f_1 \hat{e}_{2k} \hat{f}_{2k} e_1 \hat{f}_{1m} v_\lambda$. With the use of the right equalities from Proposition 3.1, we find it equal to

$$v_\lambda^* \hat{e}_{1m} f_1 \hat{e}_{2k} \hat{f}_{2k} e_1 \hat{f}_{1m} v_\lambda = [\lambda_{1m}]_q^2 v_\lambda^* \hat{e}_{2m} \hat{e}_{2k} \hat{f}_{2m} \hat{f}_{2k} v_\lambda,$$

by the induction assumption. The total contribution of this term to (7.12) is

$$[\lambda_{2k} + 1]_q^2 [\lambda_{1m}]_q^2 \prod_{j=3}^k [\lambda_{jk} + 1]_q \prod_{j=3}^k [\lambda_{jm} + 1]_q [\lambda_{k+1m} + 2]_q \prod_{j=k+2}^m [\lambda_{jm} + 1]_q C_{2k}(\lambda) C_{2m}(\lambda), \quad (7.15)$$

where again the convention about the products is in effect.

The matrix coefficient (7.12) comprises (7.13-7.15), which contain the common factor $F_1 = \prod_{j=2}^k [\lambda_{jk} + 1]_q \prod_{j=3}^k [\lambda_{jm} + 1]_q \prod_{j=k+2}^m [\lambda_{jm} + 1]_q C_{2k}(\lambda) C_{1m}(\lambda)$. Division by F_1 gives

$$[\lambda_{2k} + 1]_q [\lambda_{1m}]_q [\lambda_{k+1m} + 2]_q + [\lambda_1]_q [\lambda_{k+1m} + 1]_q [\lambda_{2k} + 1]_q [\lambda_{2m} + 1]_q - [\lambda_1]_q [\lambda_{k+1m} + 1]_q [\lambda_{2k}]_q [\lambda_{2m} + 2]_q,$$

which we denote by F_2 . The last two terms produce $[\lambda_1]_q [\lambda_{1m} - \lambda_{1k}]_q [\lambda_{k+1m} + 2]_q$ since $\lambda_{2m} - \lambda_{2m} + 1 = \lambda_{k+1m} + 2$ and $\lambda_{k+1m} + 1 = \lambda_{1m} - \lambda_{1k}$. Combine this with the first term in F_2

having made the replacement $\lambda_{2k} + 1 = \lambda_{1k} - \lambda_1$. This gives $F_2 = [\lambda_{k+1m} + 2]_q [\lambda_{1k}]_q [\lambda_{2m} + 1]_q$. Now one can see that the matrix coefficient (7.12), which is equal to $F_1 F_2$, is identical to the matrix coefficient from Example 6.9. This completes the proof. \square

In conclusion, let us turn to the contravariant Shapovalov form M_λ defined through the Chevalley involution ω . Consider the linear isomorphism $\theta: M_\lambda \rightarrow M_\lambda^*$, $\theta: uv_\lambda \mapsto v_\lambda^* \omega(u)$, where $u \in U_q(\mathfrak{n}^-)$. Obviously, $\theta(xv) = \theta(v)\omega(x)$ for all $x \in U_q(\mathfrak{g})$ and $v \in M_\lambda$. The contravariant form is defined on M_λ through the composition $M_\lambda \otimes M_\lambda \xrightarrow{\theta \otimes \text{id}} M_\lambda^* \otimes M_\lambda \rightarrow \mathbb{C}$, where the right arrow is the cyclic Shapovalov pairing.

Corollary 7.3. *The system $\frac{1}{\sqrt{B_l(\lambda)}} \hat{f}(\mathbf{l})v_\lambda$, $\mathbf{l} \in \mathfrak{T}$, forms an orthonormal basis with respect to the contravariant form on the Verma module M_λ , provided $B_l(\lambda) \neq 0, \forall \mathbf{l} \in \mathfrak{T}$.*

Proof. Follows from Theorem 6.7 and Proposition 7.2, since $\theta(\hat{f}(\mathbf{l})v_\lambda) = v_\lambda^* \hat{e}(\mathbf{l}) = v_\lambda^* \check{e}(\mathbf{l})$. \square

8 The case of $\mathfrak{g} = \mathfrak{sl}(3)$

We illustrate Theorem 6.7 on the simple example of $\mathfrak{g} = \mathfrak{sl}(3)$ reproducing the key steps of the calculations. Now $\mathfrak{n}^- = \text{Span}\{f_1, f_2, f_{12}\}$ and $\mathfrak{n}^+ = \text{Span}\{e_1, e_2, e_{12}\}$, where $f_{12} = f_1 f_2 - q f_2 f_1$ and $e_{12} = e_2 e_1 - q e_1 e_2$. The dynamical root vectors \hat{e}_{12} and \hat{f}_{12} are $\hat{e}_{12} = [h_2 + 1]_q e_2 e_1 - [h_2]_q e_1 e_2$ and $\hat{f}_{12} = f_1 f_2 [h_2 + 1]_q - f_2 f_1 [h_2]_q$. For all $l, m \in \mathbb{Z}_+$ the vector $\hat{f}_{12}^l f_1^m v_\lambda \in M_\lambda$ is annihilated by e_2 , and similarly $v_\lambda^* \hat{e}_{12}^l e_1^m \in M_\lambda^*$ is annihilated by f_2 . This readily implies

$$v_\lambda^* e_1^s \hat{e}_{12}^r e_2^p f_2^k \hat{f}_{12}^l f_1^m v_\lambda = \delta_{pk}[k]_q! \prod_{i=0}^{k-1} [\lambda_2 - i + m - l]_q v_\lambda^* \hat{e}_{12}^r e_1^s \hat{f}_{12}^l f_1^m v_\lambda$$

(we use $e_1 \hat{e}_{12} = \hat{e}_{12} e_1$, by Lemma 7.1). The matrix coefficient in the right-hand side is not zero only if $r(\alpha_1 + \alpha_2) + s\alpha_1 = l(\alpha_1 + \alpha_2) + m\alpha_1$ or, equivalently, $r = l$, $s = m$, in accordance with Proposition 5.2.

In the matrix coefficient $v_\lambda^* \hat{e}_{12}^l e_1^m \hat{f}_{12}^l f_1^m v_\lambda$, every factor $\hat{f}_{12} = [f_1, f_2]_q [h_2 + 1]_q + f_2 f_1 q^{h_2+1}$ can be replaced with $[f_1, f_2]_q [h_2 + 1]_q$. This specialization of Lemma 6.1 becomes immediate due to the fact that $f_2 f_1$ commutes with $f_1 f_2$ and can be pushed to the left, where f_2 kills $v_\lambda^* \hat{e}_{12}^l e_1^m$. This yields

$$v_\lambda^* e_1^m \hat{e}_{12}^l \hat{f}_{12}^l f_1^m v_\lambda = v_\lambda^* \hat{e}_{12}^l e_1^m \hat{f}_{12}^l f_1^m v_\lambda = \prod_{i=0}^{l-1} [\lambda_2 - i + m + 1]_q v_\lambda^* \hat{e}_{12}^l e_1^m \hat{f}_{12}^l f_1^m v_\lambda.$$

Pushing every copy of e_1 to the right produces zero contribution of the commutator $[e_1, f_{12}^l]$, as the latter belongs to $f_2 U$, as in Lemma 6.5. In the present case, this is a consequence of the commutation relations $[e_1, f_{12}] = f_2 q^{-h_i}$ and $[f_2, f_{12}]_q = 0$, see Lemma 2.1. This yields the factorization

$$v_\lambda^* \hat{e}_{12}^l e_1^m f_{12}^l f_1^m v_\lambda = v_\lambda^* \hat{e}_{12}^l f_{12}^l e_1^m f_1^m v_\lambda = [m]_q! \prod_{i=0}^{m-1} [\lambda_1 - i]_q v_\lambda^* \hat{e}_{12}^l f_{12}^l v_\lambda,$$

as in Lemma 6.5. Combining the above steps with the value of the matrix coefficient $v_\lambda^* \hat{e}_{12}^l f_{12}^l v_\lambda = [l]_q! \prod_{i=0}^{l-1} [\lambda_{12} - i]_q [\lambda_2 - i]_q$ given by (6.8) (in this simple case it can be easily computed directly) we get for $B_l(\lambda) = v_\lambda^* e_1^m \hat{e}_{12}^l e_2^k f_2^k \hat{f}_{12}^l f_1^m v_\lambda$ the formula

$$B_l(\lambda) = [l]_q! [m]_q! [k]_q! \prod_{i=0}^{k-1} [\lambda_2 - i + m - l]_q \prod_{i=0}^{l-1} [\lambda_2 - i + m + 1]_q \prod_{i=0}^{l-1} [\lambda_2 - i]_q \prod_{i=0}^{l-1} [\lambda_{12} - i]_q \prod_{i=0}^{m-1} [\lambda_1 - i]_q,$$

where $m = l_{11}$, $l = l_{12}$, and $k = l_{22}$.

In the standard basis, the inverse of Shapovalov form is known to have entries with simple poles, [16, 17]. Examining $B_l(\lambda)$ suggests the presence of second order zeros at $\lambda_2 = 0, \dots, \min\{l-1, -m+k+l-2\}$, provided l and $-m+k+l-1$ are positive. This example shows that the singularities of the form inverse are not necessarily simple in the basis $\hat{f}(\mathbf{l}), \hat{e}(\mathbf{k})$.

Consider the classical limit $q \rightarrow 1$. The set of zeros of $B_l(\lambda)$ over all $\mathbf{l} \in \mathfrak{T}$ is the union of hyperplanes $\lambda_1 \in \mathbb{Z}_+$, $\lambda_{12} \in \mathbb{Z}_+$, and $\lambda_2 \in \mathbb{Z}$. At the points $\lambda_2 \in -\mathbb{N}$ the form is still invertible, therefore the system $f_2^k \hat{f}_{12}^l f_1^m v_\lambda$ fails to be a basis. Consider the automorphism of $U(\mathfrak{sl}(3))$ corresponding to the inversion $\alpha_1 \leftrightarrow \alpha_2$ of the Dynkin diagram. This automorphism produces an alternative system of dynamical roots, with $\hat{e}_{12} = (h_1 + 1)e_1 e_2 - h_1 e_2 e_1$ and $\hat{f}_{12} = f_2 f_1 (h_1 + 1) - f_1 f_2 h_1$. With the reversed ordering on thus defined root vectors, we obtain a dynamical PBW system yielding a basis in M_λ^* and M_λ , provided $\lambda_1 \notin \mathbb{Z}$ and $\lambda_2, \lambda_{12} \notin \mathbb{Z}_+$. One or another system is a basis for $\lambda_1, \lambda_2, \lambda_{12} \notin \mathbb{Z}_+$, i.e. exactly where the Shapovalov form is non-degenerate.

9 Singular vectors in M_λ .

In this final section we use the dynamical PBW basis to construct singular vectors in M_λ .

Lemma 9.1. *Suppose $\phi_1, \phi_2 \in U_q(\mathfrak{g}_-)$ are non-zero elements of weight $-\beta \in -\mathbb{N}\Pi^+$ such that $[e_\alpha, \phi] = 0$ and $(\beta, \alpha) \neq 0$ for some $\alpha \in \Pi^+$. Then the vectors $f_\alpha \phi_1, \phi_2 f_\alpha \in U_q(\mathfrak{g}_-)$ are linearly independent.*

Proof. Put b_i the square norm of ϕ_i , $i = 1, 2$, with respect to the Shapovalov form.

First consider the case when ϕ_1 and ϕ_2 are collinear. The Gram matrix of $f_\alpha \phi_1, \phi_1 f_\alpha$ is degenerate if and only if $b_1(\lambda - \alpha)[(\lambda - \beta, \alpha)]_q - b_1(\lambda)[(\lambda, \alpha)]_q = 0$. Since $(\beta, \alpha) \neq 0$ and q is not a root of unit, $[(\lambda - \beta - m\alpha, \alpha)]_q \neq [(\lambda, \alpha)]_q$ for all $m \in \mathbb{Z}_+$. Then $b_i(\lambda)$ is divisible by $[(\lambda - \beta - m\alpha, \alpha)]_q$ for all $m \in \mathbb{Z}_+$, which is impossible as $b_i(\lambda) \neq 0$. Therefore $\{\phi_1 f_\alpha v_\lambda, f_\alpha \phi_1 v_\lambda\}$ and consequently $f_\alpha \phi_1 v_\lambda$ and $\phi_2 f_\alpha v_\lambda$ are independent for generic λ . This implies that $f_\alpha \phi_1, \phi_2 f_\alpha \in U_q(\mathfrak{g}_-)$ are linearly independent.

Now suppose that ϕ_1 and ϕ_2 are not collinear. One can assume that their Gram matrix is $\text{diag}(b_1, b_2)$, thanks to the Gram-Schmidt orthogonalization algorithm. The Gram matrix of the system $\{\phi_i f_\alpha v_\lambda, f_\alpha \phi_i v_\lambda\}_{i=1,2}$ is degenerate if and only if it is so for the subsystem $\{\phi_i f_\alpha v_\lambda, f_\alpha \phi_i v_\lambda\}$, where i is either 1 or 2. This proves independence of $\{\phi_i f_\alpha v_\lambda, f_\alpha \phi_i v_\lambda\}_{i=1,2}$ at generic λ and hence of $f_\alpha \phi_1, \phi_2 f_\alpha$. \square

Corollary 9.2. *For all $\alpha \in R^+$, the element $\hat{f}_\alpha(\lambda) \in U_q(\mathfrak{g}_-)$ is not vanishing at all λ .*

Proof. We do induction on $\deg \hat{f}_\alpha$. For $\deg \hat{f}_\alpha = 1$ the statement is obvious. Suppose that α is presentable as $\alpha = \alpha_i + \beta$, where $\alpha_i \in \Pi^+$ and $\beta \in R^+$. Suppose we have proved that $\hat{f}_\beta(\lambda) \neq 0$ for some λ . Then the non-zero vectors $f_{\alpha_i} \hat{f}_\beta v_\lambda = f_{\alpha_i} \hat{f}_\beta(\lambda) v_\lambda$ and $\hat{f}_\beta f_{\alpha_i} v_\lambda = \hat{f}_\beta(\lambda) f_{\alpha_i} v_\lambda$ are independent, by Lemma 9.1. Therefore, $\hat{f}_\alpha v_\lambda = 0$ if and only if $q^{2(\lambda+\rho, \beta)} = 1 = q^{2(\lambda+\rho, \beta)-2}$, which is impossible since $q^2 \neq 1$. \square

The standard higher root vectors $f_{ij} \in U_q(\mathfrak{g})$ are known to satisfy the identity $f_1 f_{2n}^2 = [2]_q f_{2n} f_{1n} - f_{2n}^2 f_1 = 0$, which easily follows from the Serre relations. Further we need its dynamical version.

Lemma 9.3. *One has $f_1 \hat{f}_{2n}^2 - [2]_q \hat{f}_{2n} f_1 \hat{f}_{2n} + \hat{f}_{2n}^2 f_1 = 0$.*

Proof. We prove an equivalent identity $f_1 \hat{f}_{2n}^2 [h_{2n} + 1]_q = \hat{f}_{2n}^2 f_1 [h_{2n} - 1]_q + [2]_q \hat{f}_{2n} \hat{f}_{1n}$, whose right-hand side involves ordered PBW monomials. It is clear that $f_1 \hat{f}_{2n}^2 v_\lambda$ is singular with respect to \mathfrak{g}_{3n} . Therefore, it is a linear combination of PBW monomials in $\hat{f}_{22}, \dots, \hat{f}_{2n}, \hat{f}_{12}, \dots, \hat{f}_{1n}$ applied to v_λ . By weight arguments, we can write $f_1 \hat{f}_{2n}^2 v_\lambda = A \hat{f}_{2n}^2 f_1 v_\lambda + B \hat{f}_{2n} \hat{f}_{1n} v_\lambda$ for some scalars A, B . Pairing this equality with $v_\lambda^* \hat{e}_1 \hat{e}_{2n}^2$ and $v_\lambda^* \hat{e}_{1n} \hat{e}_{2n}$ we get

$$[\lambda_1] v_\lambda^* \hat{e}_{2n}^2 \hat{f}_{2n}^2 v_\lambda = A v_\lambda^* \hat{e}_1 \hat{e}_{2n}^2 \hat{f}_{2n}^2 f_1 v_\lambda, \quad [\lambda_{1n}]_q v_\lambda^* \hat{e}_{2n}^2 \hat{f}_{2n}^2 v_\lambda = B v_\lambda^* \hat{e}_{1n} \hat{e}_{2n} \hat{f}_{2n} \hat{f}_{1n} v_\lambda,$$

where we have used Proposition 3.1 in the right equality. Comparison of the matrix coefficients yields $A = \frac{[\lambda_{2n}-1]_q}{[\lambda_{2n}+1]_q}$ and $B = \frac{[2]_q}{[\lambda_{2n}+1]_q}$, as required. \square

Corollary 9.4. Put $\bar{f}_{1n} = f_1 \hat{f}_{2n} [h_{2n} + 2]_q - \hat{f}_{2n} f_1 [h_{2n} + 1]_q$. Then $\bar{f}_{1n} \hat{f}_{2n} = \hat{f}_{2n} \bar{f}_{1n}$.

Proof. The proof readily follows from Lemma 9.3 and definition of \hat{f}_{1n} through \hat{f}_{2n} . \square

A straightforward refinement of Proposition 3.1 extends to $e_1 \hat{f}_{1n} = \hat{f}_{2n} [h_{1n}]_q + \bar{f}_{1n} e_1$. Along with Corollary 9.4, this gives

$$e_1 \hat{f}_{1n}^m = [m]_q \hat{f}_{2n} \hat{f}_{1n}^{m-1} [h_{1n} - m + 1]_q + \bar{f}_{1n}^m e_1. \quad (9.16)$$

Put formally $\hat{f}_{n+1n} = 1$. Corollary 3.1 gives rise to the following result.

Proposition 9.5. For an arbitrary weight λ and a positive integer m ,

$$e_i \hat{f}_{kn}^m v_\lambda = \delta_{ki} [m]_q [\lambda_{kn} - m + 1]_q \hat{f}_{k+1n} \hat{f}_{kn}^{m-1} v_\lambda,$$

where $i, k = 1, \dots, n$.

Proof. The delta symbol is obvious. It is then sufficient to consider the case $i = k = 1$. This is an immediate consequence of (9.16). \square

Corollary 9.2 with Proposition 9.5 gives

Corollary 9.6. The vector $\hat{f}_{kn}^m v_\lambda$ is singular if and only if $[\lambda_{kn} - m + 1]_q = 0$.

For classical universal enveloping algebras, this result was obtained in [18].

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